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Linear Algebraic Framework of Quantum State Representation

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Abstract

Linear algebra provides the mathematical foundation of quantum mechanics by rep-resenting physical states as vectors in complex Hilbert spaces and observables as linear operators. The concepts of vector spaces, basis, orthogonality, inner products, and matrix transformations enable a rigorous description of quantum systems. This paper presents a concise overview of the linear algebraic principles underlying quantum state representa-tion. The study discusses Hilbert spaces, Dirac notation, state normalization, superposition, the Born probability rule, Hermitian operators, and quantum measurement. Furthermore, the role of tensor products and matrix representations in quantum computing is briefly highlighted. An illustrative example demonstrates the application of these mathematical concepts in representing and analysing a quantum state. The paper emphasizes that linear algebra not only provides the language of quantum mechanics but also serves as the compu-tational framework for modern quantum technologies such as quantum computing, quantum information, and quantum communication.

Keywords: Linear Algebra, Quantum Mechanics, Hilbert Space, Quantum State, Hermitian Operator, Superposition, Quantum Computing.

1 Introduction

Quantum mechanics is one of the most successful scientific theories developed for describing mi-croscopic physical systems. Unlike classical mechanics, where the state of a particle is specified by deterministic quantities such as position and momentum, quantum mechanics represents a physical system by a vector in a complex Hilbert space. This mathematical representation ex-plains phenomena such as wave–particle duality, interference, superposition, and entanglement that cannot be described using classical methods.

The formulation of quantum mechanics is fundamentally based on linear algebra. Quantum states are expressed as vectors, physical observables are represented by linear operators, and measurement probabilities are obtained through inner products. Consequently, concepts such as vector spaces, orthonormal bases, eigenvalues, eigenvectors, and matrix algebra have become indispensable tools for understanding quantum theory.

The development of quantum computing has further increased the importance of linear algebra. Quantum bits (qubits), quantum gates, tensor products, and unitary transformations are all represented using matrices and vector spaces. Efficient quantum algorithms, including Shor's and Grover's algorithms, rely heavily on these mathematical structures.

This paper presents a concise mathematical overview of the linear algebraic framework used in quantum state representation. Beginning with the basic concepts of vector spaces, the discus-sion proceeds to Hilbert spaces, quantum state vectors, superposition, quantum measurement, and applications in quantum computing. The objective is to provide a clear mathematical understanding of how linear algebra forms the foundation of modern quantum mechanics.

2 Mathematical Preliminaries

The mathematical language of quantum mechanics is built upon the theory of complex vector spaces. This section briefly reviews the essential concepts that are required for representing quantum states.

2.1 Complex Vector Spaces

A complex vector space is a set of vectors together with the operations of vector addition and scalar multiplication over the field of complex numbers C.

Let V be a vector space over C. For any vectors $u, v \in V$ and scalar $\alpha \in$ C,

$$u + v \in V, \qquad \alpha v \in V.$$

Every quantum state belongs to a complex vector space, allowing linear combinations of states to form new valid states.

2.2 Linear Combination

If

$$v_1, v_2, \ldots, v_n \in V,$$

then every vector

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n,$$

where

$$a_i \in \text{C},$$

is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

In quantum mechanics, arbitrary quantum states are expressed as linear combinations of basis vectors.

2.3 Basis and Dimension

A basis of a vector space is a collection of linearly independent vectors that span the entire vector space. If

$$B = \{e_1, e_2, \ldots, e_n\},$$

is a basis of V, then every vector admits a unique representation

$$v = \sum_{i=1}^{n} c_i e_i,$$

where

$$c_i \in \mathbb{C}.$$

The number of basis vectors determines the dimension of the vector space,

$$\dim(V) = n.$$

Finite-dimensional vector spaces describe systems such as qubits, whereas infinite-dimensional Hilbert spaces model continuous quantum systems.

3 Hilbert Space and Quantum State Representation

The mathematical foundation of quantum mechanics is the concept of a Hilbert space. A Hilbert space extends the idea of a vector space by incorporating an inner product and the property of completeness. Every quantum state is represented by a normalized vector in a Hilbert space, while measurable physical quantities are represented by linear operators acting on these vectors.

3.1 Hilbert Space

A Hilbert space H is a complete inner product space over the field of complex numbers $\mathbb{C}$. For any vectors

$$|\psi\rangle,\ |\phi\rangle \in H,$$

the inner product

$$\langle\phi|\psi\rangle$$

satisfies the following properties:

1. Positivity

$$\langle\psi|\psi\rangle \geq 0.$$

2. Conjugate Symmetry

$$\langle \phi | \psi \rangle = \overline{\langle \psi | \phi \rangle}.$$

3. Linearity

$$\langle \phi | (\alpha\psi + \beta\chi) \rangle = \alpha\langle \phi | \psi \rangle + \beta\langle \phi | \chi \rangle,$$

where

$$\alpha, \beta \in \mathbb{C}.$$

The norm of a vector is defined by

$$\| \psi \| = \sqrt{\langle \psi | \psi \rangle}.$$

A valid quantum state must satisfy

$$\| \psi \| = 1.$$

3.2 Quantum State Vectors

In quantum mechanics, every physical system is represented by a state vector

$$|\psi\rangle \in H.$$

Unlike classical mechanics, which specifies a unique state using numerical variables, quantum mechanics represents the system as a vector whose direction contains complete information about the physical state.
A general quantum state can be written as

$$|\psi\rangle = \sum_{i=1}^{n} c_i |e_i\rangle,$$

where

$|e_i\rangle$ are orthonormal basis vectors and

$c_i \in \mathbb{C}$

are complex probability amplitudes satisfying

$$\sum_{i=1}^{n} |c_i|^2 = 1.$$

3.3 Dirac Bra–Ket Notation

Dirac notation provides a convenient mathematical language for quantum mechanics.
A column vector is called a **ket** and is denoted by

$$|\psi\rangle.$$

Its conjugate transpose is called a **bra**

$$\langle\psi|.$$

For example,

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix},$$

has the corresponding bra

$$\langle\psi| = \begin{bmatrix} a^* & b^* \end{bmatrix},$$

where a^* and b^* denote the complex conjugates. The inner product is written as

$$\langle\phi|\psi\rangle,$$

whereas the outer product is

$$|\psi\rangle\langle\phi|.$$

3.4 Orthonormal Basis

A set of vectors

$$\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$$

forms an orthonormal basis of the Hilbert space if

$$\langle e_i|e_j\rangle = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Every quantum state has a unique representation with respect to an orthonormal basis,

$$|\psi\rangle = \sum_{i=1}^{n} c_i|e_i\rangle.$$

The coefficients

$$c_i = \langle e_i|\psi\rangle$$

are called probability amplitudes.

3.5 Normalization of Quantum States

The normalization condition ensures that the total probability of all possible measurement outcomes is equal to one.

For the state

$$|\psi\rangle = \sum_{i=1}^{n} c_i|e_i\rangle,$$

the normalization condition is

$$\langle\psi|\psi\rangle = \sum_{i=1}^{n} |c_i|^2 = 1.$$

If a vector is not normalized, the corresponding normalized state is

$$|\psi_N\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}.$$

Example

Consider the quantum state

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle,$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The vector representation is

$$|\psi\rangle = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.$$

The normalization is verified as

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1.$$

Hence,

$|\psi\rangle$

represents a valid quantum state in the two-dimensional Hilbert space.

4 Linear Algebra in Quantum Mechanics

Linear algebra forms the mathematical language of quantum mechanics by describing quantum states as vectors and physical observables as linear operators. Concepts such as superposition, inner products, linear transformations, and Hermitian operators provide the mathematical basis for understanding quantum phenomena.

4.1 Superposition Principle

The superposition principle states that if two vectors represent valid quantum states, then any normalized linear combination of these vectors also represents a valid quantum state.
Suppose

$$|\psi_1\rangle,\ |\psi_2\rangle \in H.$$

Then

$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle,$$

where

$\alpha, \beta \in \mathbb{C},$

is also a quantum state provided

$|\alpha|^2 + |\beta|^2 = 1.$

For a two-level system,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The coefficients α and β are called probability amplitudes.

4.2 Inner Product

The inner product is one of the fundamental operations in Hilbert spaces. It determines the length of vectors, orthogonality, and the probability of measurement outcomes.
For two vectors

$$|\phi\rangle,\ |\psi\rangle \in H,$$

their inner product is

$$\langle \phi|\psi\rangle.$$

The norm of a quantum state is

$$\|\psi\| = \sqrt{\langle \psi|\psi\rangle}.$$

If

$$\langle \phi|\psi\rangle = 0,$$

then the vectors are orthogonal. For an orthonormal basis,

$$\langle e_i|e_j\rangle = \delta_{ij}.$$

4.3 Born Rule

The Born rule connects the mathematical representation of a quantum state with physical measurement probabilities.
Suppose the quantum system is in the normalized state

$$|\psi\rangle,$$

and

$$|\phi\rangle$$

is one of the basis vectors.
Then the probability of measuring the state

$$|\phi\rangle$$

is

$$P(\phi) = |\langle \phi|\psi\rangle|^2.$$

Since the basis vectors form a complete orthonormal set,

$$\sum_i |\langle e_i|\psi\rangle|^2 = 1.$$

Thus, the total probability is always conserved.

4.4 Linear Operators

In quantum mechanics, measurable physical quantities are represented by linear operators.
A mapping

$$A : H \to H$$

is called a linear operator if

$A(\alpha u + \beta v) = \alpha Au + \beta Av,$

for all

$$u, v \in H,$$

and

$\alpha, \beta \in$ C.

Examples of quantum observables represented by linear operators include

- Position
- Momentum
- Energy
- Angular momentum
- Spin

4.5 Hermitian Operators

Every measurable observable in quantum mechanics is represented by a Hermitian operator.
An operator A is Hermitian if

$A^\dagger = A,$

where $A^\dagger$ denotes the conjugate transpose of A. Hermitian operators satisfy the following properties.

1. Their eigenvalues are real.
2. Eigenvectors corresponding to different eigenvalues are orthogonal.
3. They possess a complete set of orthonormal eigenvectors.

These properties ensure that physical measurements always produce real-valued results.

4.6 Eigenvalues and Eigenvectors

The eigenvalue equation of a linear operator is

$A|\psi\rangle = \lambda|\psi\rangle,$

where

λ

is an eigenvalue and

$|\psi\rangle$

is the corresponding eigenvector. The eigenvalues are obtained from

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix.

The eigenvectors represent the possible quantum states associated with the measurement outcomes of the observable.

Example

Consider the Hermitian matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = 0,$$

which gives

$$(2 - \lambda)^2 - 1 = 0.$$

Hence, the eigenvalues are

$$\lambda_1 = 3, \qquad \lambda_2 = 1.$$

The corresponding normalized eigenvectors are

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are real and the eigenvectors are orthogonal, the matrix A is Hermitian and represents a valid quantum observable.

5 Quantum Measurement and Applications

Quantum measurement is the process through which information about a quantum system is obtained. Unlike classical systems, where measurements reveal pre-existing values, quantum measurements are probabilistic and are described using the mathematical framework of Hilbert spaces and linear operators.

5.1 Projection Operators

Projection operators describe the outcome of a quantum measurement.
For a normalized basis vector

$$|e_i\rangle,$$

the corresponding projection operator is

$$P_i = |e_i\rangle\langle e_i|.$$

The probability of obtaining the measurement outcome associated with $|e_i\rangle$ is

$$P(i) = \langle\psi|P_i|\psi\rangle = |\langle e_i|\psi\rangle|^2.$$

After measurement, the quantum state collapses to the corresponding basis vector

$$|e_i\rangle.$$

Thus, projection operators establish the mathematical relationship between quantum states and measurement outcomes.

5.2 Tensor Product Representation

Composite quantum systems are represented using tensor products of Hilbert spaces.
Suppose

$$|\psi\rangle \in H_1, \qquad |\phi\rangle \in H_2.$$

The combined quantum state is

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle.$$

For two qubits,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

their tensor product becomes

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor products are widely used in the mathematical description of quantum registers, entanglement, and multi-qubit systems.

Example

Consider the normalized quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},$$

and the observable

$$A = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}.$$

The expectation value of the observable is

$$\langle A\rangle = \langle\psi|A|\psi\rangle.$$

Since

$$A|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}2\\4\end{bmatrix},$$

we obtain

$$\langle A\rangle = \frac{1}{2}(2+4) = 3.$$

Hence, the average measurement outcome of the observable is

$$\boxed{\langle A\rangle = 3.}$$

6 Conclusion

Linear algebra provides a rigorous mathematical framework for quantum state representation. Complex vector spaces, Hilbert spaces, inner products, orthonormal bases, and linear operators together describe the behavior of quantum systems. The principles of superposition, measure-ment, and probability naturally arise from these algebraic structures.

The study also highlights the importance of Hermitian operators, tensor products, and ma-trix representations in describing quantum measurements and multi-particle systems. Further-more, the same mathematical framework supports emerging areas such as quantum computing, quantum information, and quantum communication.

Consequently, linear algebra continues to play a central role in both the theoretical under-standing and practical implementation of quantum technologies.

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