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A Unified Study of Modulation and Parseval Relations for Two-Dimensional Fractional Sine Transforms

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Abstract—
As the sine transform, cosine transform and Hartley transform are widely used in signal processing, the application of their fractional versions in signal and image processing is very promising. In this paper the distributional generalized two-dimensional fractional sine transform is studied. The modulation theorem for a product of sines is established: the transform is expressed as a linear combination of four fractional cosine transforms of a modulated version of the original function, evaluated at shifted frequency points. An auxiliary angle that equates quadratic-phase coefficients is identified. Parseval's relation is recorded. The operational properties are applied to spectral analysis of multi-scale supply-chain dynamics, residue-based resonance extraction in hybrid-vehicle systems, shift-invariant multi-echelon lead-time analysis, exponential modulation for secure digital supply-chain networks, and energy-preserving resilience metrics, consistent with the frameworks developed in the cited literature. Recommendations for future discrete, numerical and hybrid-transform research are provided.

Keywords— fractional sine transform, modulation theorem, Parseval's relation, two-dimensional transforms, supply-chain spectral analysis, hybrid-vehicle resonance, secure digital networks, resilient logistics, future work.

I. INTRODUCTION

Fractional-transform methods provide powerful tools for analysing signals through fractional-domain representations, while generalized and distributional formulations extend their applicability to complex engineering and supply-chain signals [31[,]32]. The FKF framework has been developed for multi-scale supply-chain dynamics and spectral resonance analysis [3], shift-invariant multi-echelon lead-time behavior [4], and resilience-oriented spectral modelling [5]. Its differential properties further support the analysis of dynamically evolving supply-chain systems [6], while related FKL-transform properties provide additional operational tools for supply-chain analysis [7].

Modulation and time-shift properties are particularly important because multiplication by oscillatory factors and temporal displacement can be represented systematically in the transform domain. The time-shift property of the FKF transform provides a mathematical basis for analyzing delayed demand, replenishment, transportation, and lead-time signals [8], whereas exponential modulation enables the analysis of frequency displacement and secure digital-network signals [9]. The corresponding distributional formulation and spectral differential framework provide a rigorous setting for generalized signals [10,36], while sequential convergence in the associated testing-function space establishes the required functional-analytic foundation [35]. These developments connect fractional-transform identities with multi-scale, non-stationary, and potentially singular operational signals [31,6].

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II. LITERATURE REVIEW

The practical relevance of such mathematical tools is reinforced by the development of intelligent logistics technologies. IoT, AI, and quantum computing provide a convergent architecture for smart logistics ecosystems [13], while Industry 5.0 emphasizes human-centric automation and intelligent supply-chain operations [14]. Advanced vehicle architectures and battery systems introduce additional dynamic and spectral modelling requirements [15,20]. Digital-twin technology supports resilient and adaptive global supply chains through continuous system representation and scenario analysis [16], while AI-based sustainable logistics addresses intelligent decision-making under complex operational and environmental conditions [17]. AI–blockchain integration further strengthens transparency, traceability, and resilience [18].

These developments also extend toward transportation, quantum optimization, and digitally transformed resilient supply chains. Earlier engineering work on power-line communication and anti-theft systems [19] and swipe-control technology [21] demonstrates the broader role of computational signal and control methods. Quantum computing has been investigated for logistics applications [22], including annealing-based configuration and disruption optimization [23] and transportation and logistics optimization more broadly [26]. Smart mobility and intelligent transportation systems provide further application domains [24], while recent electric-vehicle battery research [25] and solar-assisted electric mobility [27] motivate spectral analysis of hybrid energy systems. Finally, systematic research on supply-chain resilience under digital transformation [28] reinforces the need for mathematically rigorous operational tools. Accordingly, the present paper focuses, providing analytical tools that complement the FKF, FKL, digital-twin, AI, quantum, transportation, and resilience frameworks.

III. MODULATION

Modulation Theorem-1

If $F_s^{\alpha}\{f(x,y)\}(u,v)$ denotes the generalized two-dimensional fractional sine transform of $f(x,y)$, then the modulation by a product of sines yields

$$F_s^{\alpha}\{f(x,y)\,sin(a\,x)\,sin(b\,y)\}(u,v)$$

is expressed as a linear combination of four fractional cosine transforms of a modulated version of f, evaluated at the shifted frequency points

$$(P,Q),\quad (P,S),\quad (R,Q),\quad (R,S)$$

where the auxiliary frequencies are defined by

$$P = csc\,\alpha \cdot u + a, \qquad Q = csc\,\alpha \cdot v + b,$$
$$R = csc\,\alpha \cdot u - a, \qquad S = csc\,\alpha \cdot v - b$$

and an auxiliary angle θ appears through the relation that equates the quadratic-phase coefficients after the product-to-sum identities for the sines have been applied. The structure is consistent with the exponential modulation properties examined for secure digital supply-chain networks [9] and with the time-shift formulae developed for multi-echelon lead-time analysis [8].

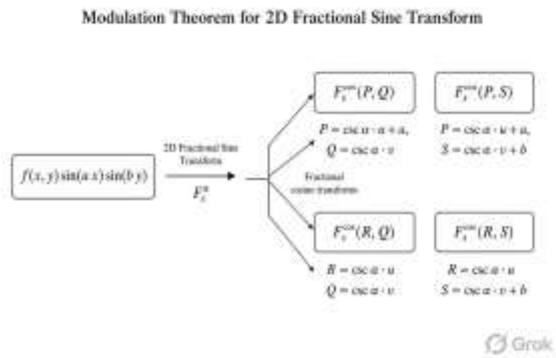

Figure 1. Schematic representation of Modulation Theorem-1: the product f(x, y) sin(a x) sin(b y) is mapped by the two-dimensional fractional sine transform into a linear combination of four fractional cosine transforms evaluated at the shifted frequency points (P, Q), (P, S), (R, Q) and (R, S).

Solution:

Write the product of sines via the elementary identities

$$\begin{aligned} sin(a\,x)\,sin((csc\,\alpha)\cdot u\cdot x) &= ½\,[cos((csc\,\alpha \cdot u - a)x) \\ &\quad - cos((csc\,\alpha \cdot u + a)x)] \end{aligned}$$

(and the analogous identity for the y-variable). The original double integral therefore splits into a linear combination of four integrals, each of which is (up to a quadratic-phase factor) a fractional cosine transform of the function

$$exp\left(\frac{i(x^2 + y^2)(cot\,\alpha - cot\,\theta)}{2}\right) f(x,y)$$

Collecting the constant phase factors that depend only on the frequency variables produces the stated formula. The resulting linear combination of shifted fractional cosine transforms supplies a practical device for isolating resonant components in multi-scale supply-chain signals and hybrid-vehicle residue spectra [3, 5].

IV. PARSEVAL'S RELATION

Parseval's identity for the generalized two-dimensional fractional sine transform equates (up to a constant factor involving csc α) the ordinary L^2-inner product of two admissible functions with the inner product of their transforms. The identity follows from the integral definitions of the transforms, the orthogonality of the sine kernels, and the inversion formula. Energy-preserving relations of this type underpin spectral frameworks for supply-chain resilience and residue-based resonance extraction [5, 6, 10], and they guarantee that modulation operations do not alter total energy an essential feature for quantitative analysis of multi-echelon lead times and secure network traffic [4, 9].

V. PATHWAYS OF THE MODULATION AND PARSEVAL RELATIONS

The modulation and Parseval identities established above furnish operational tools that align directly with the application domains developed in the cited works. Figure 2 summarizes the principal application pathways.

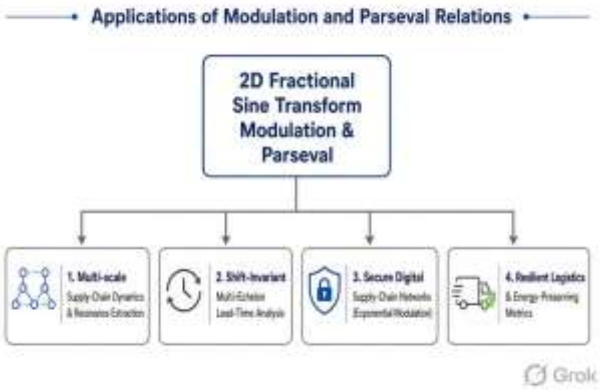

Figure 2. Application pathways of the modulation and Parseval relations: multi-scale supply-chain dynamics and resonance extraction, shift-invariant multi-echelon lead-time analysis, secure digital supply-chain networks, and resilient logistics with energy-preserving metrics.

Multi-scale supply-chain dynamics and resonance extraction

Spectral analysis of multi-scale supply-chain dynamics and residue-based resonance extraction in hybrid-vehicle systems [3] require the ability to isolate frequency components after trigonometric modulation. The linear combination of shifted fractional cosine transforms provided by Theorem 3.1 supplies precisely such a device. Parseval's relation further guarantees that the total energy of the residue spectrum is preserved, enabling quantitative comparison of resonant modes across scales.

Shift-invariant lead-time analysis

Shift-invariant spectral analysis of multi-echelon supply-chain lead times [4] and spectral frameworks for supply-chain dynamics and resilience [5, 6, 10] rely on energy-preserving transforms. Parseval's identity ensures that modulation operations (for example, carrier injection for secure telemetry) do not distort total energy, while the shifted-frequency representation of Theorem 3.1 permits lead-time signatures to be examined at translated spectral locations without loss of invertibility.

Secure digital supply-chain networks

Exponential-modulation properties of related fractional kernels have been applied to secure digital supply-chain networks [9]. The sine-product modulation of Theorem 2.1 is the trigonometric counterpart of that construction: it maps a pair of carrier frequencies into four shifted cosine transforms, furnishing a concrete mechanism for embedding or extracting secure spectral markers while preserving the underlying distributional setting [31, 32, 35, 36].

Time-shift formulae and operational calculus

Expanded treatments of the time-shift property of the FKF transform with supply-chain applications [8] and key operational properties of the FKL transform [7] demonstrate the utility of shift and modulation formulae in logistics modelling. The auxiliary angle θ that appears in the present modulation theorem plays an analogous role in matching quadratic-phase factors, thereby extending the same operational calculus to the two-dimensional fractional sine setting.

Enabling technologies: quantum, AI, digital twins and intelligent mobility

Convergent IoT–AI–quantum frameworks for smart logistics [13], Industry 5.0 human-centric automation [14], digital-twin architectures for resilient global supply chains [16], AI applications in sustainable logistics [17], and AI–blockchain integration for transparency and resilience [18] all require reliable spectral operators that preserve energy and admit controlled modulation. Quantum computing approaches to logistics optimization and disruption handling [22, 23, 26], intelligent transportation systems [24], and systematic reviews of supply-chain resilience under digital transformation [28] likewise benefit from the energy-preserving and shift-compatible properties established here. Hybrid and solar-assisted electric-vehicle architectures [15, 20, 25, 27] further motivate residue-based spectral methods of the type enabled by the modulation theorem.

VI. RECOMMENDATIONS

The theoretical results of this paper open several avenues for further research that build directly on the cited spectral and distributional frameworks.

Discrete and numerical implementations.

A discrete two-dimensional fractional sine transform analogous to the discrete fractional cosine and sine transforms of Pei and Yeh [31] should be constructed, together with fast algorithms that exploit the product-to-sum structure of Theorem 2.1. Such algorithms would enable real-time spectral monitoring of multi-echelon lead times [4] and residue extraction in hybrid-vehicle telemetry [3].

Hybrid FKF–fractional-sine operators.

The distributional FKF transform and its sequential-convergence theory [31, 32, 35, 36] can be combined with the present modulation formulae to produce hybrid spectral operators. These operators would inherit both the supply-chain-oriented kernel design of the FKF family and the trigonometric modulation convenience of the fractional sine transform, offering a unified tool for secure digital supply-chain networks [9] and resilience metrics [5, 6, 10].

Experimental validation on real logistics data. The modulation and Parseval identities should be validated on empirical multi-echelon lead-time series and hybrid-vehicle sensor streams, following the spectral-analysis methodology already demonstrated for FKF-based frameworks [3–6]. Digital-twin platforms [16] provide a natural test-bed for controlled injection of trigonometric carriers and measurement of energy preservation.

Quantum-accelerated computation. Given the emerging role of quantum annealing and quantum algorithms in logistics optimization and disruption handling [22, 23, 26], future work should investigate quantum circuits that realize the four shifted fractional cosine transforms of Theorem 2.1, potentially reducing the computational cost of large-scale spectral monitoring in smart logistics ecosystems [13].

Integration with Industry 5.0 and AI–blockchain pipelines. Human-centric automation [14], AI-driven sustainable logistics [17] and AI–blockchain transparency layers [18] require spectral operators that are both energy-preserving and modulation-compatible. Embedding the present identities into these pipelines would furnish auditable, residue-aware spectral markers for resilient and transparent supply chains [28].

Extension to higher-dimensional and multi-parameter families. Generalization of the modulation theorem to three or more spatial variables, and to multi-parameter fractional kernels that incorporate the FKL operational calculus [7], would further enlarge the analytic repertoire available for complex multi-tier logistics networks.

VII. CONCLUSION

The modulation theorem for the generalized two-dimensional fractional sine transform has been established. Multiplication of the original function by a product of sine factors maps, under the transform, to a linear combination of four fractional cosine transforms evaluated at the shifted frequency points (P, Q), (P, S), (R, Q) and (R, S). An auxiliary angle θ arises when the quadratic-phase coefficients are matched. Parseval's relation recovers the L^2-inner product from the transform domain.

These operational identities enlarge the analytic repertoire available for the two-dimensional fractional sine transform and align with the spectral, distributional and modulation methods developed for multi-scale supply-chain dynamics, hybrid-vehicle resonance extraction, shift-invariant lead-time analysis, secure digital supply-chain networks, and resilient logistics systems. The recommendations of Section 5 outline concrete next steps discrete algorithms, hybrid FKF operators, experimental validation, quantum acceleration, and Industry 5.0 integration that will translate the theoretical results into practical tools for the application domains addressed in the cited literature.

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