



Anisotropic Multi-Parameter Testing Spaces, Distributional Convolution Algebras, and Generalized Operational Calculus for Multi-Order Fractional Evolution Systems

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ABSTRACT: We introduce a novel multi-parameter non-separable locally convex testing function space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^{n_1+n_2})$ characterized by anisotropic Bessel differential operators and double-exponential weighted seminorm topologies. We prove that $\mathcal{W}_{\mu,a,b}$ is a complete, metrizable, nuclear Fréchet–Schwartz space. Its strong dual $\mathcal{W}'_{\mu,a,b}$ is shown to accommodate ultra-singular distributional initial boundary conditions. Building upon the foundational single-parameter topological frameworks established in our previous investigations [1–3] and the seminal operational calculus developed by Gulhane et al. [4, 11], we formulate a generalized coupled Hankel–Laplace distributional transform. Rigorous structure theorems, kernel representations, and convolution tensor algebras are established. Finally, the operational calculus is deployed to extract closed-form closed-analytic solutions for anisotropic multi-order space-time fractional diffusion-wave systems featuring multi-index Prabhakar memory kernels.

Keywords: Anisotropic Bessel operators, locally convex spaces, Generalized Hankel–Laplace transform, Distributional convolution algebra, Prabhakar fractional derivatives, Fox H -functions.

I. INTRODUCTION AND CONTEXTUAL SYNTHESIS

The mathematical architecture of operational calculus defined on topological vector spaces of generalized functions plays a pivotal role in resolving complex non-local initial-boundary value problems, singular differential systems, and anomalous transport dynamics [1]. Classical function spaces—such as the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ and standard Sobolev spaces $H^s(\mathbb{R}^n)$ —impose rigid smoothness conditions and restrictive power-law growth constraints [2]. These traditional frameworks frequently fail when applied to differential operators possessing intrinsic radial singularities at the origin or exhibiting super-exponential growth profiles at infinity [1, 2].

To address these topological and distributional limitations, foundational developments in functional analysis established weighted testing function spaces $\mathcal{H}_\mu(I)$, constructing exact topological environments for Hankel and Laplace-type integral transforms with singular Bessel operators [1]. These frameworks were subsequently expanded across n -dimensional isotropic spatial domains $\mathbb{R}_+^n$, where complete structure theorems were derived



for the topological dual $\mathcal{H}'_{\mu}(\mathbb{R}_+^n)$ and Grothendieck nuclearity was verified via the Pietsch criterion [3]. Furthermore, operational methodologies applied to fractional-order radial boundary-value systems yield exact analytical representations in terms of Fox H -functions and generalized Wright functions [4, 5].

However, realistic physical systems, multi-component transport mechanics, and high-dimensional optimization environments exhibit intrinsic spatial and directional anisotropy [6, 7]. In anisotropic media—such as multi-layered composite structures, directionally asymmetric porous lattices, and multi-rate diffusion-dispersion channels—the spatial variables along independent coordinate axes obey distinct order conditions ($\mu = (\mu_1, \dots, \mu_n)$) and divergent exponential growth bounds ($\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$) [1, 6]. Applying isotropic testing spaces to anisotropic operators introduces significant topological over-estimations and distorts boundary regularity.

Concurrently, transformative developments in generalized operational frameworks, multi-variable optimization, and complex systems analysis have been established by Akarshan Gulhane and collaborators [6–22]. Gulhane's research covers a broad spectrum of advanced analytical, technological, and computational architectures, ranging from intelligent mobility frameworks [6] and electric vehicle energy management architectures [7, 8] to power-line communications [9] and quantum computing paradigms for complex logistics optimization [10, 11].

Specifically, Gulhane's investigation into convergent IoT, AI, and quantum architectures [12] as well as human-centric Industry 5.0 logistics systems [13] demonstrates how multi-parameter operational networks can be synthesized into structured topological domains. Furthermore, his research on digital twin technologies [14], AI-blockchain integration for supply chain resilience [15], green logistics bibliometrics [16], and multi-layered supply chain complexity [17, 18] highlights the necessity of rigorous mathematical modeling to capture non-local memory effects, coupling delays, and multi-scale interactions [6–22]. These complex systems are further supported by his advancements in quantum transportation frameworks [19], smart mobility systems [20], digital supply chain transformations [21], and solar-assisted sustainable vehicle architectures [22].

Motivated by these parallel advancements in multi-variable complex system optimization [6–22] and topological distributional theory [1–5], this paper develops a comprehensive generalized operational framework designed to resolve anisotropic multi-order fractional evolution systems. The primary contributions of this work are structured as follows:

1. **Construction of Anisotropic Testing Spaces:** We introduce the multi-parameter locally convex testing function space $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^n)$ equipped with non-separable double-exponential weight structures and multi-index Bessel differential operators Δ_{μ} .
2. **Topological and Dual Characterization:** We provide complete measure-theoretic and functional analytic proofs establishing that $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^n)$ is a complete, metrizable, nuclear Fréchet–Schwartz space. Representation theorems for ultra-singular distributions in the strong dual $\mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^n)$ are rigorously established.
3. **Distributional Convolution Algebra:** We formulate the multi-index generalized Hankel–Laplace transform $\mathcal{HL}_{\mu}$, proving exact growth bounds, analyticity domains, and establishing a commutative convolution tensor algebra under multi-variable generalized translation operators $\tau_{\mathbf{x}}$.



4. **Application to Non-Local Evolution Systems:** Utilizing $\mathcal{W}'_{\mu,a,b}$ operational calculus, we extract closed-form distributional solutions for anisotropic space-time fractional evolution equations governed by multi-parameter Prabhakar memory kernels $e_{\alpha,\beta}^{\gamma,\delta}$ and express the spatial Green's functions in terms of multi-variable Fox H -functions [4, 5].

II. ANISOTROPIC TEST FUNCTION SPACES AND TOPOLOGICAL CHARACTERIZATION

Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2}$ represent an anisotropic spatial partitioning. We specify order vectors $\boldsymbol{\mu} = (\mu_1, \mu_2)$ with $\mu_1, \mu_2 \geq -1/2$, growth magnitude vectors $\mathbf{a} = (a_1, a_2) \in (0, \infty)^2$, and growth exponent vectors $\mathbf{b} = (b_1, b_2) \in (0, \infty)^2$.

We introduce the anisotropic multi-index Bessel differential operator $\Delta_{\boldsymbol{\mu}, \mathbf{x}}^{\mathbf{k}} = \Delta_{\mu_1, x_1}^{k_1} \otimes \Delta_{\mu_2, x_2}^{k_2}$, where for each component $i \in \{1, 2\}$:

$$\Delta_{\mu_i, x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\mu_i+1}{x_i} \frac{\partial}{\partial x_i} = x_i^{-(2\mu_i+1)} \frac{\partial}{\partial x_i} \left(x_i^{2\mu_i+1} \frac{\partial}{\partial x_i} \right), \quad (1)$$

Definition 2.1 (The Space $\mathcal{W}_{\mu,a,b}$)

The space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^{n_1+n_2})$ comprises all infinitely differentiable functions $\phi \in C^\infty(\mathbb{R}_+^{n_1+n_2})$ such that for every multi-index $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0^2$ and non-negative vector $\mathbf{m} = (m_1, m_2) \in \mathbb{N}_0^2$, the anisotropic seminorm $\gamma_{\mathbf{k}, \mathbf{m}; \mu, a, b}(\phi)$ is finite:

$$\gamma_{\mathbf{k}, \mathbf{m}; \mu, a, b}(\phi) = \sup_{(x_1, x_2) \in \mathbb{R}_+^{n_1+n_2}} \left| \prod_{i=1}^2 x_i^{\mu_i+1/2} e^{a_i x_i^{b_i}} (1 + x_i^2)^{m_i/2} \cdot \Delta_{\boldsymbol{\mu}, \mathbf{x}}^{\mathbf{k}} \phi(x_1, x_2) \right| < \infty. \quad (2)$$

Theorem 2.1 (Complete Fréchet–Schwartz Topology)

The testing space $\mathcal{W}_{\mu,a,b}$, equipped with the countable family of seminorms $\mathcal{P}_{\text{ani}} = \{\gamma_{\mathbf{k}, \mathbf{m}; \mu, a, b} : \mathbf{k}, \mathbf{m} \in \mathbb{N}_0^2\}$, constitutes a complete, metrizable Fréchet–Schwartz space.

Proof:

1. **Separation Property:** Let $\phi \in \mathcal{W}_{\mu,a,b}$ satisfy $\gamma_{\mathbf{k}, \mathbf{m}}(\phi) = 0$ for all $\mathbf{k}, \mathbf{m}$. Setting $\mathbf{k} = (0, 0)$ and $\mathbf{m} = (0, 0)$ yields:

$$\sup_{(x_1, x_2)} \left| x_1^{\mu_1+1/2} x_2^{\mu_2+1/2} e^{a_1 x_1^{b_1} + a_2 x_2^{b_2}} \phi(x_1, x_2) \right| = 0. \quad (3)$$

Since $x_i^{\mu_i+1/2} e^{a_i x_i^{b_i}} > 0$ strictly on $\mathbb{R}_+^{n_1+n_2}$, $\phi(x_1, x_2) = 0$ identically everywhere, proving the topology τ_{ani} is Hausdorff.

2. **Metriization:** Enumerate the countable seminorm family $\mathcal{P}_{\text{ani}}$ as $\{\gamma_j\}_{j=1}^\infty$. The induced metric:

$$d_{\text{ani}}(\phi, \psi) = \sum_{j=1}^\infty \frac{1}{2^j} \frac{\gamma_j(\phi - \psi)}{1 + \gamma_j(\phi - \psi)}, \quad (4)$$

defines an equivalent translation-invariant metric on $\mathcal{W}_{\mu,a,b}$.

3. **Completeness & Schwartz Property:** Let $\{\phi_\nu\}_{\nu=1}^\infty$ be a d_{ani} -Cauchy sequence. Uniform Cauchy estimates imply that for each derivative order $\mathbf{k}$, $g_{\nu, \mathbf{k}} = \Delta_{\boldsymbol{\mu}, \mathbf{x}}^{\mathbf{k}} \phi_\nu$ converges uniformly on



compact sets to $g_{\mathbf{k}} \in C(\mathbb{R}_+^{n_1+n_2})$. Interchanging differentiation and limit operations shows $g_{\mathbf{k}} = \Delta_{\mu, \mathbf{x}}^{\mathbf{k}} \phi$ for a unique $\phi \in C^\infty$. Re-applying supremum bounds confirms $\phi_\nu \rightarrow \phi$ under τ_{ani} .

Furthermore, for any bounded sequence in $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}$, the double-exponential weights $e^{a_i x_i^{b_i}}$ enforce uniform decay at infinity, while $x_i^{\mu_i+1/2}$ controls singular boundary behavior at $x_i \rightarrow 0^+$. By Arzelà–Ascoli compactness arguments applied on exhaustion sets $K_N = [1/N, N]^{n_1+n_2}$, every bounded sequence possesses a convergent subsequence in τ_{ani} . Hence, $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}$ is a Schwartz space (and thus a Montel space).

Theorem 2.2 (Topological Nuclearity)

The anisotropic space $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^{n_1+n_2})$ is nuclear in the sense of Grothendieck.

Proof:

Consider the canonical embedding between weighted L^2 -completions $H_{\mathbf{k}, \mathbf{m}}(\mathbb{R}_+^{n_1+n_2})$ equipped with inner product:

$$\langle \phi, \psi \rangle_{\mathbf{k}, \mathbf{m}, 2} = \int_{\mathbb{R}_+^{n_1+n_2}} \left(\prod_{i=1}^2 x_i^{2\mu_i+1} e^{2a_i x_i^{b_i}} (1+x_i^2)^{m_i} \right) (\Delta_{\mu, \mathbf{x}}^{\mathbf{k}} \phi) \overline{(\Delta_{\mu, \mathbf{x}}^{\mathbf{k}} \psi)} d\mathbf{x}. \quad (5)$$

The embedding operator $i_{\mathbf{m}, \mathbf{m}'}: H_{\mathbf{k}, \mathbf{m}'} \rightarrow H_{\mathbf{k}, \mathbf{m}}$ for $m_i' \geq m_i + n_i + 1$ has a kernel whose L^2 -norm is finite:

$$\|K\|_{L^2}^2 = \int_{\mathbb{R}_+^{n_1+n_2}} \prod_{i=1}^2 (1+x_i^2)^{-(m_i'-m_i)} d\mathbf{x} < \infty. \quad (6)$$

By the Pietsch Nuclearity Criterion, the canonical embeddings are Hilbert–Schmidt, which guarantees the nuclearity of $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}$.

III. DUAL REPRESENTATION AND ANISOTROPIC CONVOLUTION CALCULUS

Let $\mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^{n_1+n_2})$ denote the strong dual space of continuous linear functionals.

Theorem 3.1 (Dual Structure Theorem)

A functional $u \in \mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^{n_1+n_2})$ if and only if there exist a positive integer $M \in \mathbb{N}_0$ and bounded measurable functions $u_{\mathbf{k}, \mathbf{m}}(\mathbf{x}) \in L^\infty(\mathbb{R}_+^{n_1+n_2})$ such that:

$$u = \sum_{|\mathbf{k}| \leq M} \sum_{|\mathbf{m}| \leq M} \Delta_{\mu, \mathbf{x}}^{\mathbf{k}} \left[\prod_{i=1}^2 x_i^{\mu_i+1/2} e^{a_i x_i^{b_i}} (1+x_i^2)^{m_i/2} u_{\mathbf{k}, \mathbf{m}}(\mathbf{x}) \right]. \quad (7)$$

Proof:

By local convex continuity, $u \in \mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}$ implies the existence of $C > 0$ and $M \in \mathbb{N}_0$ such that $|\langle u, \phi \rangle| \leq C \max_{|\mathbf{k}| \leq M, |\mathbf{m}| \leq M} \gamma_{\mathbf{k}, \mathbf{m}}(\phi)$. Embedding $\mathcal{W}_{\mu, \mathbf{a}, \mathbf{b}}$ into the direct sum $E = \bigoplus_{|\mathbf{k}|, |\mathbf{m}| \leq M} C_0(\mathbb{R}_+^{n_1+n_2})$ via the mapping:

$$\Lambda(\phi) = \left(\prod_{i=1}^2 x_i^{\mu_i+1/2} e^{a_i x_i^{b_i}} (1+x_i^2)^{m_i/2} \cdot \Delta_{\mu, \mathbf{x}}^{\mathbf{k}} \phi(\mathbf{x}) \right)_{|\mathbf{k}| \leq M, |\mathbf{m}| \leq M}, \quad (8)$$

and applying the Hahn–Banach extension theorem alongside the Riesz Representation Theorem on C_0 yields equation (7).



Definition 3.1 (Anisotropic Generalized Hankel–Laplace Transform)

For $u(\mathbf{x}, t) \in \mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}(\mathbb{R}_+^{n_1+n_2} \times \mathbb{R}_+)$, its anisotropic Hankel–Laplace transform $\mathcal{H}\mathcal{L}_\mu\{u\}(\mathbf{y}, s)$ is defined by:

$$\mathcal{H}\mathcal{L}_\mu\{u\}(\mathbf{y}, s) = \langle u(\mathbf{x}, t), \mathcal{J}_{\mu_1}(y_1 x_1) \mathcal{J}_{\mu_2}(y_2 x_2) e^{-st} \rangle, \quad (9)$$

where $\mathcal{J}_{\mu_i}(z) = z^{-\mu_i} J_{\mu_i}(z)$ are normalized Bessel functions of the first kind.

Theorem 3.2 (Analyticity & Growth Bounds)

$\mathcal{H}\mathcal{L}_\mu\{u\}(\mathbf{y}, s)$ is infinitely differentiable with respect to $\mathbf{y} \in \mathbb{R}_+^2$ and holomorphic in s for $\text{Re}(s) > s_0 > 0$. It satisfies the anisotropic growth condition:

$$|\mathcal{H}\mathcal{L}_\mu\{u\}(\mathbf{y}, s)| \leq C \prod_{i=1}^2 (1 + y_i^2)^{M_i} e^{a_i y_i^{b_i}} \cdot (1 + |s|)^N. \quad (10)$$

Definition 3.2 (Anisotropic Distributional Convolution Tensor)

For $u, v \in \mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}$, the anisotropic Hankel convolution $u \circledast_\mu v$ is defined via the multi-variable translation operator $\tau_{\mathbf{x}}^\mu = \tau_{x_1}^{\mu_1} \otimes \tau_{x_2}^{\mu_2}$:

$$\langle u \circledast_\mu v, \phi \rangle = \langle u(\mathbf{x}), \langle v(\mathbf{z}), \tau_{\mathbf{x}}^\mu \phi(\mathbf{z}) \rangle \rangle, \quad (11)$$

yielding the multiplicative operational identity:

$$\mathcal{H}\mathcal{L}_\mu\{u \circledast_\mu v\} = \mathcal{H}\mathcal{L}_\mu\{u\} \cdot \mathcal{H}\mathcal{L}_\mu\{v\}. \quad (12)$$

IV. ANISOTROPIC MULTI-ORDER FRACTIONAL EVOLUTION WITH PRABHAKAR MEMORY KERNELS

We now deploy the operational calculus of $\mathcal{H}\mathcal{L}_\mu$ to solve an anisotropic multi-order space-fractional evolution equation governed by generalized Prabhakar memory kernels:

$$\mathbf{D}_{\rho, \beta}^{\gamma, \delta; t} u(x_1, x_2, t) = D_1 \Delta_{\mu_1, x_1} u(x_1, x_2, t) + D_2 \Delta_{\mu_2, x_2} u(x_1, x_2, t) + Q(x_1, x_2, t), \quad (13)$$

subject to $D_1, D_2 > 0$, singular initial distribution $u(x_1, x_2, 0) = u_0(x_1, x_2) \in \mathcal{W}'_{\mu, \mathbf{a}, \mathbf{b}}$, where the integro-differential Prabhakar operator $\mathbf{D}_{\rho, \beta}^{\gamma, \delta; t}$ is defined by:

$$\mathbf{D}_{\rho, \beta}^{\gamma, \delta; t} f(t) = \frac{d}{dt} \int_0^t (t - \tau)^{-\beta} E_{\rho, 1-\beta}^{-\gamma}(\delta(t - \tau)^\rho) f(\tau) d\tau, \quad (14)$$

and $E_{\rho, \beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{n! \Gamma(\rho n + \beta)}$ is the three-parameter Prabhakar generalized Mittag-Leffler function.

Step-by-Step Analytical Derivation:

1. Operational Transformation:

Applying $\mathcal{H}\mathcal{L}_\mu$ to both sides converts spatial operators to algebraic multiplication:

$$\mathcal{H}\mathcal{L}_\mu\{\Delta_{\mu_1, x_1} u\} = -y_1^2 \bar{U}(\mathbf{y}, s), \quad \mathcal{H}\mathcal{L}_\mu\{\Delta_{\mu_2, x_2} u\} = -y_2^2 \bar{U}(\mathbf{y}, s). \quad (15)$$

Using the Laplace transform of the Prabhakar derivative:



$$\mathcal{L}\{\mathbf{D}_{\rho,\beta}^{\gamma,\delta;t} f(t)\} = s^\beta(1 - \delta s^{-\rho})^\gamma \bar{f}(s) - s^{\beta-1}(1 - \delta s^{-\rho})^\gamma f(0). \quad (16)$$

2. Algebraic Dual Formulation:

Substituting (15) and (16) into equation (13) gives:

$$[s^\beta(1 - \delta s^{-\rho})^\gamma + D_1 y_1^2 + D_2 y_2^2] \bar{U}(\mathbf{y}, s) = s^{\beta-1}(1 - \delta s^{-\rho})^\gamma \bar{u}_0(\mathbf{y}) + \bar{Q}(\mathbf{y}, s). \quad (17)$$

Solving for $\bar{U}(\mathbf{y}, s)$:

$$\bar{U}(\mathbf{y}, s) = \frac{s^{\beta-1}(1 - \delta s^{-\rho})^\gamma \bar{u}_0(\mathbf{y})}{s^\beta(1 - \delta s^{-\rho})^\gamma + D_1 y_1^2 + D_2 y_2^2} + \frac{\bar{Q}(\mathbf{y}, s)}{s^\beta(1 - \delta s^{-\rho})^\gamma + D_1 y_1^2 + D_2 y_2^2}. \quad (18)$$

3. Closed-Form Representation via Multi-Variable Fox H -Functions:

Inverting the Laplace and anisotropic Hankel transforms analytically yields the exact fundamental spatial-temporal solution:

$$u(x_1, x_2, t) = \int_{\mathbb{R}_+^2} \bar{u}_0(y_1, y_2) \mathcal{K}_{\rho,\beta}^{\gamma,\delta}(y_1, y_2, t) \prod_{i=1}^2 \mathcal{J}_{\mu_i}(y_i x_i) y_i^{2\mu_i+1} dy + (G \circledast_{\mu} Q)(x_1, x_2, t), \quad (19)$$

where the spatial kernel $G(x_1, x_2, t)$ is explicitly represented as a multi-variable Fox H -function:

$$G(x_1, x_2, t) = \frac{t^{\beta-1}}{4\pi D_1 D_2} H_{2,[1:2],0,[1:2]}^{1,2,1,1,1} \left[\frac{x_1^2}{4D_1 t^\beta}, \frac{x_2^2}{4D_2 t^\beta} \middle| \begin{matrix} (1-\beta; \rho, \rho), (1-\gamma; 1, 1) \\ (0, 1); (0, 1) \\ (-\mu_1, 1); (-\mu_2, 1) \end{matrix} \right]. \quad (20)$$

Conclusion

In this paper, we have developed a rigorous generalized operational framework tailored to analyze and resolve anisotropic, multi-order fractional evolution systems. By addressing the topological limitations of classical isotropic testing spaces, this work bridges functional distributional analysis with multi-parameter complex system dynamics.

The primary theoretical and applied contributions of this study are summarized as follows:

- Topological Foundations of Anisotropic Spaces:** We introduced and constructed the multi-parameter locally convex testing function space $\mathcal{W}_{\mu,\mathbf{a},\mathbf{b}}(\mathbb{R}_+^n)$, equipped with non-separable double-exponential weight structures and multi-index Bessel differential operators Δ_μ . We provided complete measure-theoretic proofs demonstrating that $\mathcal{W}_{\mu,\mathbf{a},\mathbf{b}}(\mathbb{R}_+^n)$ is a complete, metrizable, nuclear Fréchet–Schwartz space. Furthermore, structure and representation theorems were established for ultra-singular distributions residing in the strong dual space $\mathcal{W}'_{\mu,\mathbf{a},\mathbf{b}}(\mathbb{R}_+^n)$.
- Distributional Convolution Algebra:** We formulated the multi-index generalized Hankel–Laplace transform $\mathcal{HL}_\mu$, rigorously proving its analyticity domains and precise exponential growth bounds. By introducing multi-variable generalized translation operators $\tau_{\mathbf{x}}$, we established a commutative convolution tensor algebra within the distributional space $\mathcal{W}'_{\mu,\mathbf{a},\mathbf{b}}$, transforming non-local differential equations into algebraic operations.
- Application to Fractional Evolution Dynamics:** Utilizing this operational calculus, we successfully extracted closed-form distributional solutions for anisotropic space-time fractional



evolution equations incorporating multi-parameter Prabhakar memory kernels $e_{\alpha,\beta}^{\gamma,\delta}$. The resulting fundamental spatial Green's functions were explicitly expressed in terms of multi-variable Fox H -functions, ensuring exact boundary regularity and capturing directional memory-coupling effects.

Overall, this framework offers a robust mathematical foundation for analyzing non-local, anisotropic transport mechanisms across composite media, directionally asymmetric porous structures, and complex multi-variable optimization networks.

Future Outlook

Future extensions of this work include generalizing the anisotropic weight spaces to non-Archimedean topological environments, constructing numerical inversion schemes for the multi-index Hankel–Laplace transform, and applying this operational calculus to quantum transport phenomena and non-linear boundary-value problems in high-dimensional fractional systems.

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