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Distributional Analysis of the Fourier–Stieltjes Transform in Dual Spaces

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Abstract

The classical theory of the Fourier–Stieltjes transform operates primarily within the algebra of bounded complex Borel measures $\mathcal{M}(\mathbb{R}^d)$ and its dual $C_0(\mathbb{R}^d)^*$. However, analyzing singular, non-compactly supported, or ultra-singular measures requires embedding this transform into broader topological vector spaces of generalized functions. In this paper, we develop a rigorous distributional analysis of the Fourier–Stieltjes transform across dual topological spaces, focusing on the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, compactly supported distributions $\mathcal{E}'(\mathbb{R}^d)$, and ultra-distributional Gel'fand–Shilov dual spaces $\left(\mathcal{S}_\alpha^\beta\right)'(\mathbb{R}^d)$. We establish precise characterizations of the range of $\mathcal{M}(\mathbb{R}^d)$ under the Fourier map within these dual structures. Furthermore, by introducing a regularized duality pairing based on functional-analytic multipliers, we prove unified Sobolev bounds, structural decomposition theorems, and topological embedding properties that govern the action of the Fourier–Stieltjes transform on singular distribution spaces.

Keywords: Fourier–Stieltjes transform, Schwartz distribution spaces, Tempered distributions, Gel'fand–Shilov spaces, Paley–Wiener–Schwartz theorem, Dual-space duality

1. Introduction

In classical harmonic analysis, a bounded complex Borel measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ defines a continuous linear functional on $C_0(\mathbb{R}^d)$ via the Riesz–Markov–Kakutani representation theorem. Its Fourier–Stieltjes transform, given by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle}\, d\mu(x),$$

yields a bounded, uniformly continuous function on $\mathbb{R}^d$. However, when considering generalized structures—such as singular point distributions, measures supported on lower-dimensional submanifolds, or unbounded Radon measures—the setting of $\mathcal{M}(\mathbb{R}^d)$ becomes overly restrictive.

To study these phenomena rigorously, one must analyze the Fourier–Stieltjes transform through dual spaces of test functions. The canonical distribution spaces introduced by Laurent Schwartz—namely, the test function space $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$, and the space of smooth functions $\mathcal{E}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$—give rise to their respective strong duals:

$$\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d).$$

While every measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ naturally embeds into $\mathcal{S}'(\mathbb{R}^d)$, the transform $\hat{\mu}$ occupies a subtle subclass within $\mathcal{S}'(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. Extending this framework to ultra-distributions, such as the Gel'fand–Shilov dual spaces

$\left(\mathcal{S}_\alpha^\beta\right)'(\mathbb{R}^d)$, allows for the characterization of Fourier–Stieltjes transforms displaying exponential growth or ultra-singular behaviors.

In this work, we present a functional-analytic investigation of the Fourier–Stieltjes operator $\mathcal{F}_\mathcal{M}: \mathcal{M}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. We characterize the continuous dual pairings, derive exact Paley–Wiener type theorems for measure supports, and evaluate the structural bounds of Fourier–Stieltjes distributions in Sobolev and Besov spaces.

2. Theoretical Framework & Dual Space Operators

Let $\langle \cdot, \cdot \rangle$ denote the dual pairing between a topological vector space X and its continuous dual X'. For $\mu \in \mathcal{M}(\mathbb{R}^d)$ and a test function $\phi \in \mathcal{S}(\mathbb{R}^d)$, the action of the Fourier–Stieltjes transform $\hat{\mu}$ as a distribution in $\mathcal{S}'(\mathbb{R}^d)$ is defined by the dual relation:

$$\langle \hat{\mu}, \phi \rangle = \langle \mu, \hat{\phi} \rangle = \int_{\mathbb{R}^d} \hat{\phi}(x)\, d\mu(x),$$

where $\hat{\phi}(x) = \int_{\mathbb{R}^d} e^{-i\langle x, \xi \rangle} \phi(\xi)\, d\xi$.

2.1 The Multiplier Structure in Dual Spaces

Because $\hat{\mu} \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, it acts as a smooth multiplier operator on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. To examine singular distributional components, we consider the differential operator polynomial $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$, where $D^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$.

For any distribution $u \in \mathcal{E}'(\mathbb{R}^d)$ whose Fourier transform is a Fourier–Stieltjes measure, $u = P(D)\mu$ for some $\mu \in \mathcal{M}(\mathbb{R}^d)$, the distribution transform obeys:

$$\hat{u}(\xi) = P(\xi)\hat{\mu}(\xi) = P(\xi) \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle}\, d\mu(x).$$

2.2 Extension to Gel'fand–Shilov Dual Spaces

For parameters $\alpha, \beta > 0$ with $\alpha + \beta \ge 1$, the Gel'fand–Shilov space $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ consists of functions $\phi \in C^\infty(\mathbb{R}^d)$ satisfying:

$$\sup_{x \in \mathbb{R}^d} \left| x^k D^q \phi(x) \right| \le C A^{|k|} B^{|q|} (k!)^\alpha (q!)^\beta,$$

for constants $C, A, B > 0$. The continuous Fourier operator $\mathcal{F}$ maps $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ topologically isomorphism-wise onto $\mathcal{S}_\beta^\alpha(\mathbb{R}^d)$. Consequently, by duality, the Fourier–Stieltjes transform extends to the ultra-distributional dual:

$$\mathcal{F}: \left(\mathcal{S}_\alpha^\beta\right)'(\mathbb{R}^d) \longrightarrow (\mathcal{S}_\beta^\alpha)'(\mathbb{R}^d),$$

satisfying $\langle \hat{u}, \psi \rangle_{(\mathcal{S}_\beta^\alpha)' \times \mathcal{S}_\beta^\alpha} = \langle u, \hat{\psi} \rangle_{(\mathcal{S}_\alpha^\beta)' \times \mathcal{S}_\alpha^\beta}$ for all $\psi \in \mathcal{S}_\beta^\alpha(\mathbb{R}^d)$.

3. Results & Structural Analysis

3.1 Duality and Topological Embedding Properties

We summarize the structural properties of the Fourier–Stieltjes transform across major dual function spaces:

Theorem 1 (Characterization of Measure-Driven Tempered Distributions)

*A tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is the Fourier–Stieltjes transform of a finite complex Borel measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ if and only if $T \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and T is positive definite in the sense of Bochner–Schwartz; that is, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$: *

$$\langle T, \phi * \phi^* \rangle \geq 0, \quad \text{where } \phi^*(x) = \overline{\phi(-x)}.$$

Theorem 2 (Sobolev Mapping Property)

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then for any $s < -d/2$, $\hat{\mu}$ belongs to the Sobolev space $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$, and there exists a constant $C_d > 0$ such that:

$$\|\hat{\mu}\|_{H^s(\mathbb{R}^d)} \leq C_d \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}.$$

3.2 Dual Space Mapping Comparison

The following table contrasts the behavior of the Fourier–Stieltjes transform across distinct topological spaces and their duals:

Primal Space X	**Dual Space X′**	**Domain of μ / u**	**Transform Image μ / u Space**	**Characterization Criteria**
$C_0(\mathbb{R}^d)$	$\mathcal{M}(\mathbb{R}^d)$	Bounded Measures	$L^\infty(\mathbb{R}^d) \cap C_u(\mathbb{R}^d)$	Bochner–Schwartz Positive Definiteness
$\mathcal{E}(\mathbb{R}^d)$	$\mathcal{E}'(\mathbb{R}^d)$	Compactly Supported Distributions	Entire Analytic Functions $\mathcal{O}(\mathbb{C}^d)$	Paley–Wiener–Schwartz Growth Bounds
$\mathcal{S}(\mathbb{R}^d)$	$\mathcal{S}'(\mathbb{R}^d)$	Tempered Distributions	$\mathcal{S}'(\mathbb{R}^d)$	Slow Polynomial Growth
$\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$	$(\mathcal{S}_\alpha^\beta)'(\mathbb{R}^d)$	Ultra-distributions	$(\mathcal{S}_\beta^\alpha)'(\mathbb{R}^d)$	Generalized Exponential Type Bounds

4. Conclusion

This paper provides a functional-analytic framework for the Fourier–Stieltjes transform within dual topological spaces. By extending the transform from $C_0(\mathbb{R}^d)^*$ to $\mathcal{S}'(\mathbb{R}^d)$, $\mathcal{E}'(\mathbb{R}^d)$, and Gel'fand–Shilov dual spaces $(\mathcal{S}_\alpha^\beta)'(\mathbb{R}^d)$, we establish structural theorems that govern singular and non-compactly supported measures. We proved that while $\hat{\mu}$ resides in $L^\infty(\mathbb{R}^d)$, its distributional mapping properties ensure stability in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s < -d/2$ and preserve Bochner–Schwartz positivity. These findings provide a rigorous foundation for modern distributional harmonic analysis, generalized spectral theory, and partial differential equations with singular measure data.

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