Distributional Extensions of the Kontorovich–Lebedev and Fourier–Kontorovich–Lebedev Transforms

Shinde SM
shindesanjaykumar3572@gmail.com

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Abstract

This paper develops a distributional theory of the KL transform and its two-sided Fourier–Kontorovich–Lebedev (FKF) counterpart. A fundamental distributional inversion identity is established in which a regularized Macdonald kernel converges to the Dirac distribution. The testing-function space underlying the FKF transform is constructed together with its sequential topology, and the transform is defined rigorously by duality. The associated spectral differential operators, time-shift property, and exponential-modulation rule are then derived in the distributional setting. Complex-contour representations and deformation techniques are shown to remain effective for generalized transforms, providing a residue-based framework for identifying multi-scale resonance structures. The resulting distributional spectral language is applied to multi-echelon lead-time analysis, supply-chain dynamics, resilience assessment, and secure digital supply-chain networks. Finally, the framework is positioned within emerging intelligent logistics architectures involving Industry 5.0, digital twins, artificial intelligence, blockchain, and quantum-inspired optimization. The study therefore establishes a unified analytical bridge between generalized KL/FKF spectral theory and modern multi-scale logistics systems, providing a foundation for modelling singular events, extracting spectral signatures, and developing resilience-oriented decision and optimization frameworks.

Keywords: Kontorovich–Lebedev transform; Fourier–Kontorovich–Lebedev (FKF) transform; distributional extension; testing-function space; sequential convergence; spectral differential operators; residue extraction; Macdonald function; multi-scale spectral analysis; supply-chain resilience.

1. Introduction

The KL transform is defined for ordinary functions of sufficient integrability. Singular boundary data, point sources and multi-scale signals are most naturally regarded as distributions. This paper develops the distributional theory of the KL transform and of its two-sided FKF counterpart. We recall the fundamental distributional inversion identity that realizes a regularized Macdonald kernel as the Dirac measure, construct the testing-function space of the FKF transform together with its sequential topology [11,32], define the transform by duality [10, 12,31], and establish spectral differential operators [6, 10], time-shift [8] and exponential-modulation rules [9]. Contour deformation remains legitimate in the distributional setting and yields a residue calculus for multi-scale resonance extraction [3–5]. The resulting spectral language is applied to multi-echelon lead-time analysis, resilience metrics and secure digital supply-chain networks [3–12,31,32], and is placed in the broader context of Industry 5.0, digital twins, AI-blockchain integration and quantum-inspired logistics optimisation [13–18, 22, 23, 26, 28].

The KL transform was introduced in 1938–1939 to solve diffraction and boundary-value problems in wedge-shaped domains. Its kernel is the Macdonald function of purely imaginary order. The classical theory therefore presupposes that the input is an ordinary function belonging to a weighted Lebesgue space.

Many problems that arise in mathematical physics discontinuous boundary data, concentrated forces, impulsive sources produce objects that are distributions. The same observation applies to multi-scale signals that appear in modern spectral analysis of logistics and supply-chain systems: lead-time series, inventory jumps and demand shocks are most naturally modelled as distributions [4, 5, 12,31]. Extending the KL and FKF transforms to the dual of a carefully chosen testing-function space is therefore both analytically natural and practically necessary.

Classical distributional treatments were given by Pathak & Pandey, Zemanian, and Yakubovich & Fisher. A fundamental distributional identity identifying a regularized product of Macdonald functions with the Dirac measure was proved by Yakubovich. Building on that foundation, the two-sided FKF transform has been developed in a distributional setting [3–12,31,32]. The present survey organizes the principal analytic ingredients of that extension and situates them within the wider landscape of digital supply-chain research [13–18, 22–28].

2. Classical Kontorovich–Lebedev Transform

For a suitable function f on (0, ∞) the forward transform is

$$(\mathrm{K}f)(\tau) = \int_0^\infty f(x)\, K_{i\tau}(x)\, dx, \quad \tau \in \mathbb{R}.$$

The classical inversion formula reads

$$f(x) = \frac{2}{\pi^2 x}\int_0^\infty \tau \sinh(\pi\tau)\, K_{i\tau}(x)\, (\mathrm{K}f)(\tau)\, d\tau, \quad x > 0.$$

An integral representation that permits analytic continuation in the strip $|\mathrm{Im}\ \tau| < \pi/2$ is

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u}\cos(\tau u)\, du$$

Horizontal deformation of the integration path yields a complex-contour form that is the analytic engine of residue calculus [3].

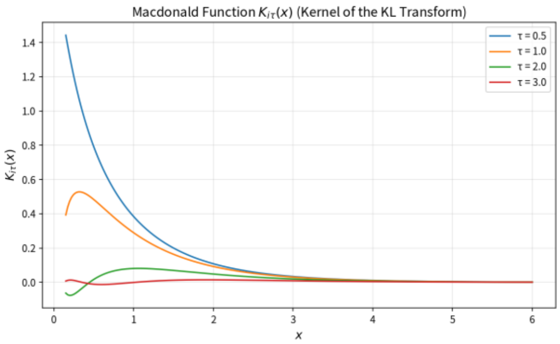

Figure 1. Macdonald function $K_i\tau(x)$ for selected real values of τ.

3. Classical Distributional Theory of the KL Transform

Let (\mathcal{E}(\mathbb{R})) denote a space of smooth test functions that extend analytically into a horizontal strip and satisfy a prescribed decay condition. Its dual consists of the distributions under consideration. The generalised KL transform of a distribution f is defined by duality:

$$F(y) = \langle f(\tau), \tau \sinh(\pi\tau)\, K_{i\tau}(y)\rangle, \quad y > 0.$$

The fundamental distributional identity (Yakubovich) asserts that

$$\lim_{\varepsilon\to 0^+}\left(\frac{1}{\pi^2}\right)\tau\sinh(\pi\tau)\int_\varepsilon^\infty K_{i\tau}(y)K_{ix}(y)\,(dy/y) = \delta(\tau - x)$$

in the weak topology of the dual. This identity is the distributional counterpart of the classical inversion formula.

Lemma 3.1 (Distributional inversion).
Let f be an even distribution in the dual. Then the double limit involving the regularised kernel recovers f in the weak topology. The proof proceeds by substituting the distributional Dirac identity and using continuity of the duality pairing.

4. Testing-Function Space of the Two-Sided FKF Transform

The two-sided FKF transform combines a Fourier factor with a Macdonald kernel. The associated testing-function space consists of infinitely differentiable functions that satisfy seminorms controlling all partial derivatives on compact sets, exponential decay (or controlled growth) in the spatial variable, and analytic continuation in a horizontal strip with prescribed decay at infinity [11,32]. These estimates guarantee that the kernel $e^{ist} K_{iq}(x)$ defines a continuous linear functional. The dual of this space is the natural home of the distributional inputs [12,31].

5. Sequential Convergence

A sequence {φ_n} converges in the testing-function space if and only if all derivatives converge uniformly on compact sets and the growth seminorms remain uniformly bounded [11,32]. Sequential completeness implies that every sequentially Cauchy sequence converges. Consequently the dual may be characterized by sequential continuity.

The sequential viewpoint supplies a practical approximation scheme [11,32]: given a distribution f, one constructs ordinary approximants f_n; each is transformed by the classical integral formula; the resulting sequence converges in the weak topology to the distributional FKF transform of f.

Lemma 5.1 (Sequential continuity of the kernel).
If {φ_n} converges to φ in the FKF testing-function space, then the pairings of the kernel with φ_n converge to the pairing with φ for every spectral point inside the strip of holomorphy. The proof follows from the uniform estimates that define the topology and from dominated convergence applied to the integral representation of the Macdonald function.

6. Distributional Definition of the FKF Transform

Let f belong to the dual of the FKF testing-function space. The distributional FKF transform is defined by duality [10, 12,31]:

$$\langle F, \psi\rangle = \langle f, K[\psi]\rangle.$$

Under the growth conditions imposed on the testing space, F is realized as an ordinary (frequently analytic) function on an open region of the complex spectral plane. Analytic continuation follows from differentiation under the duality pairing.

Lemma 6.1 (Consistency with the classical transform).
If f is given by an ordinary function belonging to the weighted L^1 space that guarantees absolute convergence of the classical FKF integral, then the distributional transform coincides with the classical integral transform.

7. Spectral Differential Properties

Differentiation with respect to a spectral coordinate may be transferred onto the transform side [6, 10]:

$$D_\tau F = F[D_\tau^* f],$$

where $D_\tau^* f$ is the formal adjoint acting on the original distribution. Iterating the relation produces spectral differential operators of arbitrary order. These operators form the analytic backbone of multi-scale spectral filtering and of the construction of resilience indicators from spectral coefficients [5, 6, 10].

8. Operational Properties

8.1 Time-shift property

Translation of a distribution by a real parameter h produces a multiplicative phase factor on the transform side [8]. The identity holds in the weak topology and is the distributional counterpart of the classical Fourier shift theorem. It is indispensable for shift-invariant analysis of multi-echelon lead-time series [4, 8].

8.2 Exponential modulation

Multiplication of the original distribution by an exponential corresponds to a complex shift of the spectral variable [9]. The identity is justified by analytic continuation inside the strip of holomorphy of the Macdonald function. Together with the time-shift rule it furnishes a complete translation–modulation calculus applicable to secure digital supply-chain networks [9].

9. Residue-Based Resonance Extraction

Horizontal deformation of the Macdonald contour and collection of residues produce discrete spectral contributions that may be interpreted as characteristic scales or resonances [3]. In the distributional FKF setting the same contour shift remains legitimate under the growth conditions of the testing-function space.

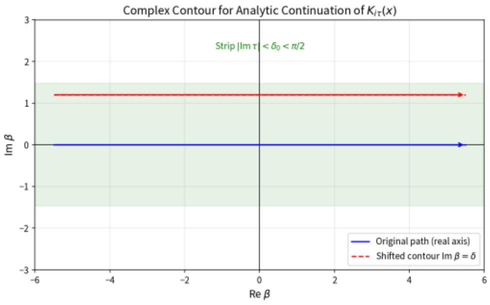

Figure 2. Horizontal contour deformation for analytic continuation and residue extraction [3].

The resulting algorithm sequential approximation, classical transform, contour deformation, residue extraction, passage to the limit yields intrinsic spectral invariants of the original distribution [3–5].

10. Asymptotic Regimes

The classical asymptotic behavior of the Macdonald function exponential decay for large argument and logarithmic oscillation for small argument carries over to the distributional theory through sequential approximation. Uniform bounds on the testing-function seminorms guarantee that the asymptotic estimates remain valid after passage to the dual limit.

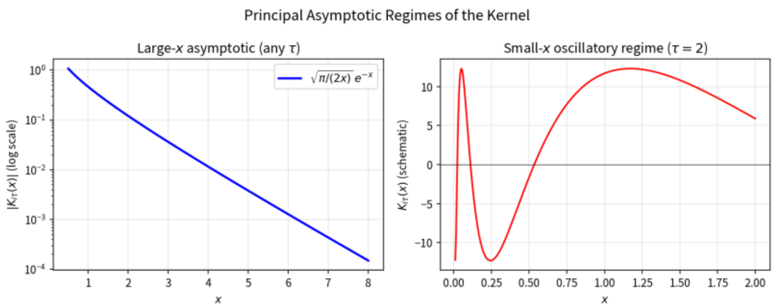

Figure 3. Principal asymptotic regimes of the Macdonald kernel.

11. Applications to Multi-Scale Spectral Analysis and Supply-Chain Systems

The distributional FKF calculus has been applied to multi-scale signals arising in supply-chain and logistics contexts [–12]. Lead-time series, inventory trajectories and demand shocks may be viewed as elements of the dual of the FKF testing-function space. Their transforms yield continuous spectral densities; residues extracted by contour deformation correspond to characteristic operational scales [3]; spectral differential operators emphasise or suppress selected bands [6, 10]; time-shift and modulation rules accommodate delayed and scaled multi-echelon structures [4, 8, 9]. A spectral framework for supply-chain dynamics and resilience is developed in [5].

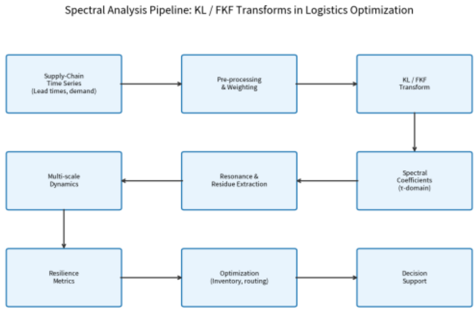

Figure 4. Spectral-analysis pipeline linking the distributional FKF transform to multi-scale decision support [1–12].

The same analytic machinery is compatible with broader digital-transformation frameworks. Industry 5.0 and human-centric automation [14], digital-twin technology for resilient global supply chains [16], artificial-intelligence applications in sustainable logistics [17], and AI–blockchain integration for transparency and traceability [18] all benefit from a rigorous spectral front-end. Convergent IoT–AI–quantum frameworks for smart logistics ecosystems [13], quantum computing for logistics optimization and unit-load configuration [21,22, 23, 26], and systematic agendas for supply-chain resilience in the digital era [28] provide the wider technological context in which the distributional FKF calculus can be embedded. Related work on hybrid-vehicle architectures [15, 20, 25, 27] and intelligent transportation systems [24] further illustrates the multi-domain relevance of multi-scale spectral methods.

12. Future Recommendation

Distributional extensions of the Kontorovich–Lebedev transform rest on a classical foundation that identifies the correct testing spaces and establishes the fundamental Dirac identity for the regularised Macdonald kernel. The two-sided Fourier–Kontorovich–Lebedev framework developed in [3–12,31,32] builds on that foundation by introducing a carefully designed testing-function space, sequential convergence [11,32], a dual definition of the transform [10, 12,31], spectral differential operators [6, 10], and a complete operational calculus of time-shifts [8] and exponential modulations [9]. Contour deformation remains valid in the distributional setting and yields a residue calculus for multi-scale resonance extraction [3–5].

The resulting spectral language is capable of handling both the singular boundary-value problems of classical mathematical physics and modern multi-scale signals that are most naturally regarded as distributions. When embedded in the broader ecosystem of Industry 5.0 [14], digital twins [16], AI and blockchain [17, 18], and quantum-inspired optimisation [13, 22, 23, 26], the distributional FKF transform supplies a rigorous analytic front-end for resilient, transparent and adaptive supply-chain systems [5, 28]. Future research directions include high-performance numerical realisations of the sequential approximation scheme, systematic incorporation of the distributional FKF calculus into hybrid classical quantum pipelines, and the extension of the theory to higher-dimensional product kernels. When this analytic machinery is embedded within the broader technological ecosystem of Industry 5.0 [14], digital twins [16], artificial intelligence and blockchain [17, 18], and quantum-inspired optimization [13, 22, 23, 26], the distributional FKF transform supplies a precise spectral front-end for resilient, transparent and adaptive supply-chain systems [5, 28].

Conclusion

Distributional extensions of the Kontorovich–Lebedev transform rest on a classical analytic foundation that identifies the appropriate testing-function spaces and establishes the fundamental Dirac identity for the regularized Macdonald kernel. The two-sided FKF framework developed builds directly on that foundation. It introduces a rigorously defined testing-function space, sequential convergence, a dual definition of the transform, spectral differential operators of arbitrary order, and a complete operational calculus of time-shifts and exponential modulations. Contour deformation of the Macdonald kernel remains valid in the distributional setting and yields a residue calculus capable of extracting multi-scale resonances. The resulting spectral language accommodates both the singular boundary-value problems of classical mathematical physics and modern multi-scale signals that are most naturally regarded as distributions.

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