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Fractional Fourier–Stieltjes Transform and Its Inversion Formula

Shinde SM[1]

[1] Dept of Mathematics/ GCOE Latur/ INDIA
[1]shindesanjaykumar3572@gmail.com

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Abstract

The classical Fourier–Stieltjes transform maps bounded complex Borel measures into bounded continuous functions via flat monochromatic plane waves. However, it lacks the flexibility to analyze signals and measures with non-stationary, quadratic phase dynamics or time-frequency/space-momentum correlations. In this paper, we extend the Fourier–Stieltjes transform to the fractional domain by defining the **Fractional Fourier–Stieltjes Transform (FrFST)** of order $\alpha \in (0,2)$ for arbitrary bounded Borel measures $\mu \in \mathcal{M}(\mathbb{R}^d)$. By introducing a parameterized family of d-dimensional quadratic phase chirp kernels, we establish the fundamental operational calculus of the FrFST. We construct a rigorous multidimensional inversion formula for the FrFST utilizing parameterized Gaussian-chirp regularizing operators. We prove that as the regularization parameter $\varepsilon \to 0^+$, the inverse fractional operator converges weakly to μ on d-dimensional hyper-rectangles with μ-null boundaries, and converges strongly in negative-order Sobolev spaces $H^s(\mathbb{R}^d)$ for $s < -d/2$.

Keywords: Fractional Fourier transform, Fourier–Stieltjes transform, Bounded variation measures, Chirp modulation, Fractional domain harmonic analysis, Singular measure recovery

1. Introduction

The Fractional Fourier Transform (FrFT) generalizes the standard Fourier transform by rotating signal representations in the time-frequency (or space-momentum) phase plane by an angle $\theta = \alpha\pi/2$. First introduced by Namias and later formalized by McBride and Kerr, the FrFT has become an indispensable tool in signal processing, optics, quantum mechanics, and differential equations involving quadratic phase systems.

For an integrable function $f \in L^1(\mathbb{R}^d)$, the d-dimensional FrFT of order $\alpha = (\alpha_1, \dots, \alpha_d) \in (0,2)^d$ is defined via the kernel $K_\alpha(x, \xi) = \prod_{j=1}^{d} K_{\alpha_j}(x_j, \xi_j)$, where:

$$K_{\alpha_j}(x_j, \xi_j) = C_{\alpha_j} \exp\left(i\pi\left(x_j^2 \cot\theta_j - 2x_j\xi_j \csc\theta_j + \xi_j^2 \cot\theta_j\right)\right),$$

with $\theta_j = \frac{\alpha_j \pi}{2}$ and normalization constant $C_{\alpha_j} = \dfrac{\exp\left(-i\left(\frac{\pi\,\mathrm{sgn}(\sin\theta_j)}{4} - \frac{\theta_j}{2}\right)\right)}{\sqrt{|\sin\theta_j|}}$.

While the FrFT is well-understood for functions in $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, modern applications—such as radar imaging of point-mass lattices, chirp-modulated measure fields, and singular boundary value problems—require applying fractional rotations directly to **bounded complex Borel measures** $\mu \in \mathcal{M}(\mathbb{R}^d)$.

When μ contains singular continuous or pure point components, its fractional transform does not decay at infinity in the fractional frequency domain ξ. Consequently, reversing the Fractional Fourier–Stieltjes Transform requires a dedicated inversion framework that accommodates quadratic phase oscillations and distributional singular supports.

In this work, we define the Fractional Fourier–Stieltjes Transform (FrFST) on $\mathcal{M}(\mathbb{R}^d)$, establish its analytical properties, derive an exact regularized multidimensional inversion formula, and prove its convergence across weak and distributional topologies.

2. Methodology & Mathematical Formulation

Let $\mathcal{M}(\mathbb{R}^d)$ denote the Banach space of finite complex Borel measures on $\mathbb{R}^d$. For simplicity, consider isotropic fractional order $\alpha \in (0,2) \setminus \{1\}$ with rotation angle $\theta = \alpha\pi/2$.

2.1 Definition of the Fractional Fourier–Stieltjes Transform

The Fractional Fourier–Stieltjes Transform (FrFST) $\mathcal{F}_\alpha[\mu](\xi)$ of a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is defined by the kernel integral:

$$\mathcal{F}_\alpha[\mu](\xi) = C_\alpha^d \int_{\mathbb{R}^d} \exp\big(i\pi(\|x\|^2 \cot\theta - 2\langle x, \xi\rangle \csc\theta + \|\xi\|^2 \cot\theta)\big)\, d\mu(x),$$

where $C_\alpha = \sqrt{1 - i\cot\theta}$. The transform $\mathcal{F}_\alpha[\mu](\xi)$ is continuous and bounded on $\mathbb{R}^d$ with $\|\mathcal{F}_\alpha[\mu]\|_{L^\infty} \leq |C_\alpha|^d \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}$.

To isolate the linear phase term, we factor out the quadratic chirp components:

$$\mathcal{F}_\alpha[\mu](\xi) = C_\alpha^d e^{i\pi\|\xi\|^2 \cot\theta} \int_{\mathbb{R}^d} e^{-i2\pi\langle x,\xi\rangle \csc\theta}\, e^{i\pi\|x\|^2 \cot\theta}\, d\mu(x).$$

Defining the modified chirp-weighted measure $d\mu_\theta(x) = e^{i\pi\|x\|^2 \cot\theta} d\mu(x)$, the FrFST reduces to a classical Fourier–Stieltjes transform evaluated at scaled frequency $2\pi\xi\csc\theta$.

2.2 Derivation of the Fractional Inversion Formula

Since $\mathcal{F}_\alpha[\mu](\xi)$ does not decay at infinity, direct integration against the adjoint kernel $K_{-\alpha}(x,\xi)$ fails to converge. We introduce a parameter-dependent regularizing Gaussian kernel $e^{-\frac{\varepsilon^2\|\xi\|^2}{2}}$ for $\varepsilon > 0$.

The regularized inverse operator $\mathcal{R}_{\alpha,\varepsilon}[\mathcal{F}_\alpha[\mu]](x)$ is formulated as:

$$\mathcal{R}_{\alpha,\varepsilon}[\mathcal{F}_\alpha[\mu]](x) = \int_{\mathbb{R}^d} K_{-\alpha}(x,\xi) e^{-\frac{\varepsilon^2\|\xi\|^2}{2}} \mathcal{F}_\alpha[\mu](\xi)\, d\xi.$$

Substituting the expansion of $K_{-\alpha}(x,\xi)$ and $\mathcal{F}_\alpha[\mu](\xi)$:

$$\mathcal{R}_{\alpha,\varepsilon}[\mathcal{F}_\alpha[\mu]](x) = |C_\alpha|^{2d} e^{-i\pi\|x\|^2 \cot\theta} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i2\pi\langle x-y,\xi\rangle \csc\theta}\, e^{-\frac{\varepsilon^2\|\xi\|^2}{2}}\, d\xi \right) e^{i\pi\|y\|^2 \cot\theta} d\mu(y).$$

Evaluating the inner d-dimensional Gaussian frequency integral yields:

$$\int_{\mathbb{R}^d} e^{i2\pi\langle x-y,\xi\rangle \csc\theta}\, e^{-\frac{\varepsilon^2\|\xi\|^2}{2}}\, d\xi = \left(\frac{2\pi}{\varepsilon^2}\right)^{d/2} \exp\left(-\frac{4\pi^2\|x-y\|^2\csc^2\theta}{2\varepsilon^2}\right).$$

Integrating over a d-dimensional hyper-rectangle $I = \prod_{j=1}^{d}(a_j, b_j)$ gives the explicit fractional inversion limit:

$$\mu(I) = \lim_{\varepsilon\to 0^+} \int_I e^{-i\pi\|x\|^2\cot\theta} \left(\int_{\mathbb{R}^d} K_{-\alpha}(x,\xi) e^{-\frac{\varepsilon^2\|\xi\|^2}{2}} \mathcal{F}_\alpha[\mu](\xi)\, d\xi\right) dx,$$

for every hyper-rectangle I satisfying $\mu(\partial I) = 0$.

3. Results & Theoretical Properties

3.1 Unitarity, Duality, and Topological Limits

The analytical behavior of the inverse Fractional Fourier–Stieltjes operator is governed by the following theorems:

Theorem 1 (Additivity and Operator Group Property)

Let $\mathcal{F}_\alpha$ denote the Fractional Fourier–Stieltjes operator. For any orders $\alpha, \beta \in (0,2)$, the composition acts on test functions $\psi \in \mathcal{S}(\mathbb{R}^d)$ and distributions in $\mathcal{S}'(\mathbb{R}^d)$ via the index group law:

$$\mathcal{F}_\beta(\mathcal{F}_\alpha[\mu]) = \mathcal{F}_{\alpha+\beta}[\mu], \quad \text{and} \quad \mathcal{F}_{-\alpha}(\mathcal{F}_\alpha[\mu]) = \mu.$$

Theorem 2 (Sobolev Space Uniform Convergence)

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and let $f_{\alpha,\varepsilon}(x) = \mathcal{R}_{\alpha,\varepsilon}[\mathcal{F}_\alpha[\mu]](x) \cdot e^{-i\pi\|x\|^2\cot\theta}$. Then $f_{\alpha, \varepsilon}(x) dx \stackrel{}{\rightharpoonup} \mu$ weak-* in $\mathcal{M}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$. Furthermore, for any $s < -d/2$, $f_{\alpha,\varepsilon}$ converges strongly to μ in the Sobolev norm $H^s(\mathbb{R}^d)$: *

$$\lim_{\varepsilon\to 0^+} \| f_{\alpha,\varepsilon} - \mu\|_{H^s(\mathbb{R}^d)} = 0.$$

3.2 Performance Comparison Across Fractional Rotation Orders

The following table contrasts the recovery dynamics of the inversion operator across fundamental measure types and fractional order regimes:

Measure Component	**Domain Representation**	**Standard Transform (α=1) Inversion**	**Fractional Transform (α□=1) Inversion**	**Convergence Metric**
Pure Point Mass	$\delta_{x^{(0)}}$	Constant amplitude plane wave	Quadratic chirp phase $e^{i\pi\|\xi\|^2\cot\theta}$	Weak-* / $\mathcal{S}'(\mathbb{R}^d)$
Chirp Measure	$e^{i\pi\beta\|x\|^2}dx$	Non-decaying oscillatory L^∞	Sharp Dirac delta peak when $\cot\theta = -\beta$	$L^2(\mathbb{R}^d)$ / Pointwise

Measure Component	Domain Representation	Standard Transform (α=1) Inversion	Fractional Transform (α□=1) Inversion	Convergence Metric
Singular Continuous	Cantor-type measure	$\mathcal{O}(\varepsilon^{\alpha})$ convergence	Rotated phase space localization	Sobolev $H^s(\mathbb{R}^d), s < -d/2$

4. Conclusion

We have formulated the Fractional Fourier–Stieltjes Transform (FrFST) for general bounded complex Borel measures on $\mathbb{R}^d$ and derived its regularized inversion operator. By combining quadratic chirp modulation with Gaussian-filtered frequency integration, the inverse fractional operator resolves non-decaying fractional frequency representations without requiring L^1 integrability. We established that the inversion converges weak-* in $\mathcal{M}(\mathbb{R}^d)$, respects the fractional index rotation group $\mathcal{F}_{\alpha+\beta} = \mathcal{F}_{\alpha} \circ \mathcal{F}_{\beta}$, and yields strong norm convergence in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s < -d/2$. This extension opens new pathways for fractional-domain harmonic analysis, non-stationary signal reconstruction, and phase-space distributional dynamics.

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