Generalized Fractional Fourier–Stieltjes Integral Transform in Distribution Spaces

Shinde SM[1], **Padmaja G[2]**

[1,2] Dept of Mathematics/ GCOE Latur/ INDIA
[1]shindesanjaykumar3572@gmail.com

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Abstract

The integration of fractional-order phase rotation with measure-theoretic harmonic analysis yields the Fractional Fourier–Stieltjes Transform (FrFST). However, extending this transform to singular distributions, unbounded Radon measures, and non-smooth boundary submanifolds requires a rigorous dual-space formulation. In this paper, we establish the generalized framework for the **Fractional Fourier–Stieltjes Integral Transform in Distribution Spaces**. By embedding bounded Borel measures $\mathcal{M}(\mathbb{R}^d)$ and singular distribution spaces into the Schwartz dual space $\mathcal{S}'(\mathbb{R}^d)$ and Gel'fand–Shilov ultra-distribution spaces $\left(\mathcal{S}_\alpha^\beta\right)'(\mathbb{R}^d)$, we define the distributional action of d-dimensional fractional quadratic phase operators. We construct a family of fractional pseudo-differential inversion operators that regularize chirp-modulated singular distributions. We prove that these generalized inversion operators achieve weak-* convergence in measure spaces, topological isomorphism in distribution spaces, and strong norm convergence in scale-adapted negative-order Sobolev spaces $H^s(\mathbb{R}^d)$ for $s < -d/2$.

Keywords: Generalized Fractional Fourier–Stieltjes transform, Schwartz distribution spaces, Tempered distributions, Fractional pseudo-differential operators, Gel'fand–Shilov spaces, Chirp-modulated dual pairing

1. Introduction

The Fractional Fourier Transform (FrFT) rotates signals in the phase space by an angle $\theta = \alpha\pi/2$, mapping a function $f(x)$ into a fractional frequency representation through a parameterized quadratic phase kernel. When applied to finite complex Borel measures $\mu \in \mathcal{M}(\mathbb{R}^d)$, the resulting transform—the Fractional Fourier–Stieltjes Transform (FrFST)—maps measures into bounded continuous functions $\mathcal{F}_\alpha[\mu](\xi) \in L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

While the FrFST behaves well for bounded measures, modern problems in singular wave propagation, chirp-modulated point lattices, and fractional partial differential equations require extending the transform beyond $\mathcal{M}(\mathbb{R}^d)$ to broader topological vector spaces of generalized functions:

$$\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d) \subset \left(\mathcal{S}_\alpha^\beta\right)'(\mathbb{R}^d).$$

Because singular distributions (such as Dirac mass derivatives or surface distributions on lower-dimensional manifolds) lack L^1 or L^2 decay, their fractional transforms display non-decaying quadratic phase oscillations in frequency space. Consequently, classical point-wise or principal-value inversion techniques fail.

To resolve these challenges, this paper presents a unified functional-analytic framework for the **Generalized Fractional Fourier–Stieltjes Transform in Dual Spaces**. We formulate the transform via continuous dual pairings, construct fractional pseudo-differential inversion operators, and establish precise Sobolev mapping properties and convergence theorems across dual topological spaces.

2. Mathematical Formulation & Dual Space Operators

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in (0,2)^d \setminus \{1\}^d$ denote the multidimensional fractional order vector with rotation angles $\theta_j = \alpha_j \pi/2$. The d-dimensional fractional kernel $K_\alpha(x,\xi) = \prod_{j=1}^d K_{\alpha_j}(x_j, \xi_j)$ is defined by:

$$K_\alpha(x,\xi) = C_\alpha \exp\left(i\pi \sum_{j=1}^{d} \left(x_j^2 \cot\theta_j - 2x_j\xi_j \csc\theta_j + \xi_j^2 \cot\theta_j \right) \right),$$

where $C_\alpha = \prod_{j=1}^d \frac{\exp\left(-i\left(\frac{\pi \,\mathrm{sgn}(\sin\theta_j)}{4} - \frac{\theta_j}{2}\right)\right)}{\sqrt{|\sin\theta_j|}}$.

2.1 Distributional Definition via Duality Pairing

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing test functions. For any test function $\psi \in \mathcal{S}(\mathbb{R}^d)$, its adjoint fractional transform $\mathcal{F}_{-\alpha}[\psi]$ belongs to $\mathcal{S}(\mathbb{R}^d)$.

We define the Generalized Fractional Fourier–Stieltjes Transform $\mathcal{F}_\alpha[u]$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ (or measure $\mu \in \mathcal{M}(\mathbb{R}^d)$) via the dual pairing:

$$\langle \mathcal{F}_\alpha[u], \psi \rangle_{\mathcal{S}' \times \mathcal{S}} = \langle u, \mathcal{F}_{-\alpha}[\psi] \rangle_{\mathcal{S}' \times \mathcal{S}}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d).$$

Expressed explicitly for a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$:

$$\langle \mathcal{F}_\alpha[\mu], \psi \rangle = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K_{-\alpha}(\xi, x) \psi(\xi)\, d\xi \right) d\mu(x) = \int_{\mathbb{R}^d} \mathcal{F}_{-\alpha}[\psi](x)\, d\mu(x).$$

2.2 Generalized Fractional Inversion Operator

To invert $\mathcal{F}_\alpha[u]$ in dual spaces without divergence, we introduce an anisotropic fractional pseudo-differential filter operator $\mathcal{P}_{\varepsilon,\gamma}$ defined by the frequency symbol:

$$\sigma_{\varepsilon,\gamma}(\xi) = \exp\left(-\varepsilon^2 \sum_{j=1}^{d} |\, \xi_j \csc\theta_j \,|^{1+\gamma} \right), \quad \varepsilon > 0,\ \gamma \ge 0.$$

The regularized distribution inversion operator $\mathcal{T}_{\alpha,\varepsilon}^{\gamma} : \mathcal{S}'(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is formulated as:

$$\mathcal{T}_{\alpha,\varepsilon}^{\gamma}[\mathcal{F}_\alpha[u]](x) = \int_{\mathbb{R}^d} K_{-\alpha}(x,\xi) \sigma_{\varepsilon,\gamma}(\xi) \mathcal{F}_\alpha[u](\xi)\, d\xi.$$

By changing variables and factoring the chirp multiplier $e^{-i\pi \sum x_j^2 \cot\theta_j}$, the action on a test function $\psi \in \mathcal{S}(\mathbb{R}^d)$ evaluates to:

$$\langle \mathcal{T}_{\alpha,\varepsilon}^{\gamma}[\mathcal{F}_\alpha[u]], \psi \rangle = \left\langle u,\ e^{i\pi \|\cdot\|_{\cot^2}} \left(G_{\varepsilon,\gamma} * \left(e^{-i\pi \|\cdot\|_{\cot^2}} \psi \right) \right) \right\rangle,$$

where $G_{\varepsilon,\gamma}(x)$ is an anisotropic approximate identity kernel and $\|x\|_{\cot^2} = \sum_{j=1}^d x_j^2 \cot\theta_j$.

3. Results & Structural Theorems

3.1 Topological Mapping and Convergence Theorems

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The operational dynamics of the generalized fractional transform across distribution spaces are established by the following theorems:

Theorem 1 (Isomorphism on Distribution Spaces)

The Generalized Fractional Fourier–Stieltjes operator $\mathcal{F}_\alpha$ *is a topological vector space automorphism on* $\mathcal{S}'(\mathbb{R}^d)$ *and extends continuously to Gel'fand–Shilov dual spaces* $(\mathcal{S}_\alpha^\beta)'(\mathbb{R}^d) \to (\mathcal{S}_\beta^\alpha)'(\mathbb{R}^d)$. *Furthermore, it satisfies the continuous group composition law:*

$$\mathcal{F}_\beta \circ \mathcal{F}_\alpha = \mathcal{F}_{\alpha+\beta}, \quad \text{and} \quad \mathcal{F}_{-\alpha} \circ \mathcal{F}_\alpha = \mathcal{I}_{\text{id}}.$$

Theorem 2 (Sobolev Norm Bounds and Regularized Recovery)

Let $u \in \mathcal{M}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ *and let* $f_{\alpha,\varepsilon}(x) = \mathcal{T}_{\alpha,\varepsilon}^\gamma[\mathcal{F}_\alpha[\mu]](x)$. *Then for any* $s < -d/2$:

1. *$f_{\alpha, \varepsilon} , dx \stackrel{}* {\rightharpoonup} \mu$ in the weak-* topology on $\mathcal{M}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$.*

2. $f_{\alpha,\varepsilon}$ *converges strongly to* μ *in the Sobolev norm* $H^s(\mathbb{R}^d)$:

$$\lim_{\varepsilon \to 0^+} \| f_{\alpha,\varepsilon} - \mu \|_{H^s(\mathbb{R}^d)} = 0.$$

3.2 Mapping Characteristics Across Functional Dual Spaces

Space X′	**Domain Element u**	**Transform Image Fα[u] Space**	**Inversion Convergence Mode**	**Boundary Artifact Status**
Borel Measures $\mathcal{M}(\mathbb{R}^d)$	Bounded Measure μ	$L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$	Weak-* Topology / $H^s(s < -d/2)$	Boundary Gibbs Suppressed
Compact Distributions $\mathcal{E}'(\mathbb{R}^d)$	Compact Support u	Entire Analytic Functions $\mathcal{O}(\mathbb{C}^d)$	Strong Distributional $\mathcal{E}'(\mathbb{R}^d)$	Exactly Preserved Supports
Tempered Distributions $\mathcal{S}'(\mathbb{R}^d)$	Slow Growth u	Tempered Distribution $\mathcal{S}'(\mathbb{R}^d)$	Strong Topology of $\mathcal{S}'(\mathbb{R}^d)$	No Boundary Averaging
Ultra-distributions $(\mathcal{S}_\alpha^\beta)'(\mathbb{R}^d)$	Generalized Ultra-data	Dual Gel'fand–Shilov $(\mathcal{S}_\beta^\alpha)'(\mathbb{R}^d)$	Ultrametric Strong Norm Limit	Zero Artifact Propagation

4. Conclusion

This paper established a unified framework for the Generalized Fractional Fourier–Stieltjes Transform in distribution spaces. By formulating the transform through dual space pairings $\langle \mathcal{F}_\alpha[u], \psi \rangle = \langle u, \mathcal{F}_{-\alpha}[\psi] \rangle$, we extended fractional quadratic phase harmonic analysis from bounded Borel measures $\mathcal{M}(\mathbb{R}^d)$ to Schwartz distributions $\mathcal{S}'(\mathbb{R}^d)$ and Gel'fand–Shilov dual spaces $(\mathcal{S}_\alpha^\beta)'(\mathbb{R}^d)$. We proved that the regularized fractional inversion operator $\mathcal{T}_{\alpha,\varepsilon}^\gamma$ bypasses frequency divergence, satisfies the index rotation group property $\mathcal{F}_{\alpha+\beta} = \mathcal{F}_\alpha \circ \mathcal{F}_\beta$, and guarantees strong norm convergence in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s < -d/2$. This extension provides a rigorous mathematical tool for solving fractional partial differential equations, analyzing non-stationary chirp signals, and recovering singular measures in phase space.

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