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Generalized Inversion Formula for the Multidimensional Fourier–Stieltjes Transform

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Abstract

The Fourier–Stieltjes transform extends classical Fourier analysis to bounded complex Borel measures on $\mathbb{R}^d$. While inversion formulas for standard $L^1(\mathbb{R}^d)$ functions are well established via pointwise or mean convergence, reconstructing a general measure μ from its Fourier–Stieltjes transform $\hat{\mu}(\xi)$ presents fundamental measure-theoretic challenges, particularly at points or hypersurfaces of discontinuity. In this paper, we establish a generalized multidimensional inversion formula for the Fourier–Stieltjes transform applicable to arbitrary bounded Borel measures on $\mathbb{R}^d$. By introducing a parameterized family of regularizing kernels based on Gaussian and spherical summation operators, we construct a limit process that recovers the measure value on d-dimensional intervals (hyper-rectangles) whose boundary has zero μ-measure. Furthermore, we extend this framework to tempered distributions, showing that our formula unifies the singular, absolutely continuous, and discrete components of μ under a singular integral limit. The structural properties of the inversion operator are rigorously analyzed, yielding explicit bounds and convergence rates in the weak-* topology.

Keywords: Fourier–Stieltjes transform, Multidimensional inversion formula, Bounded variation measures, Schwartz distributions, Harmonic analysis, Bochner's theorem

1. Introduction

Fourier analysis serves as a foundational pillar in pure and applied mathematics, providing crucial tools for partial differential equations, probability theory, and signal processing. When transitioning from integrable functions in $L^1(\mathbb{R}^d)$ to general finite complex Borel measures $\mathcal{M}(\mathbb{R}^d)$, the classical Fourier transform generalizes to the Fourier–Stieltjes transform. For a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, its Fourier–Stieltjes transform $\hat{\mu}: \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle}\, d\mu(x),$$

where $\langle \xi, x\rangle = \sum_{j=1}^{d} \xi_j\, x_j$ denotes the standard Euclidean inner product on $\mathbb{R}^d$.

The function $\hat{\mu}(\xi)$ is bounded and uniformly continuous on $\mathbb{R}^d$, but unlike transforms of L^1 functions, $\hat{\mu}(\xi)$ does not necessarily vanish at infinity (as shown by discrete point-mass measures). Consequently, reversing this transform to recover μ from $\hat{\mu}$ cannot be achieved through direct L^1 integration of $e^{i\langle \xi, x\rangle}\hat{\mu}(\xi)$.

In one dimension, classic results such as Levy's continuity theorem and Gil-Pelaez's inversion formula resolve this issue for cumulative distribution functions by evaluating integrals over symmetric limits. However, expanding these techniques to $d > 1$ dimensions introduces geometric complexities. High-dimensional boundaries, non-rectangular geometry, and anisotropic singular supports require a unified framework capable of handling all measure components:

$$\mu = \mu_{\text{ac}} + \mu_{\text{sing}} + \mu_{\text{pp}},$$

representing the absolutely continuous, singularly continuous, and pure point parts, respectively.

In this work, we present a complete generalized inversion formula for $\hat{\mu}(\xi)$ in $\mathbb{R}^d$. By analyzing regularized kernel operators of the Fejér and Gauss–Weierstrass types, we establish a precise limit formula that evaluates $\mu(I)$ for any bounded hyper-rectangle $I = \prod_{j=1}^{d}(a_j, b_j)$ satisfying $\mu(\partial I) = 0$. We prove weak-* convergence of the regularized measure densities to μ and extend the formulation to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

2. Methodology & Mathematical Formulation

To reconstruct $\mu \in \mathcal{M}(\mathbb{R}^d)$ from $\hat{\mu}(\xi)$, we construct an approximate identity via a smooth, rapidly decreasing kernel $\Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\Phi(0) = 1$. For $\varepsilon > 0$, define the scaled kernel $\Phi_\varepsilon(x) = \varepsilon^{-d}\Phi(x/\varepsilon)$ with Fourier transform $\widehat{\Phi}(\varepsilon\xi)$.

2.1 The Regularized Measure Density

We define the regularized function $f_\varepsilon(x)$ on $\mathbb{R}^d$ by convolving μ with Φ_ε:

$$f_\varepsilon(x) = (\Phi_\varepsilon * \mu)(x) = \int_{\mathbb{R}^d} \Phi_\varepsilon\,(x - y)\, d\mu(y).$$

Applying the inverse Fourier transform to the kernel integration yields:

$$f_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle}\, \widehat{\Phi}(\varepsilon\xi)\hat{\mu}(\xi)\, d\xi.$$

To isolate the measure evaluated on an open hyper-rectangle $I = \prod_{j=1}^{d}(a_j, b_j)$, we introduce the characteristic indicator function $\chi_I(x)$. Integrating $f_\varepsilon(x)$ over I gives:

$$\mu_\varepsilon(I) = \int_I f_\varepsilon\,(x)\, dx = \frac{1}{(2\pi)^d} \int_I \left(\int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle}\, \widehat{\Phi}(\varepsilon\xi)\hat{\mu}(\xi)\, d\xi \right) dx.$$

By Fubini's theorem, interchanging the order of integration yields:

$$\mu_\varepsilon(I) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\int_I e^{i\langle \xi, x\rangle}\, dx \right) \widehat{\Phi}(\varepsilon\xi)\hat{\mu}(\xi)\, d\xi.$$

Evaluating the inner integral over the hyper-rectangle I:

$$\int_I e^{i\langle \xi, x\rangle}\, dx = \prod_{j=1}^{d} \frac{e^{i\xi_j b_j} - e^{i\xi_j a_j}}{i\xi_j}.$$

2.2 Generalized Inversion Formula

Using the product kernel $\widehat{\Phi}(\varepsilon\xi) = \prod_{j=1}^{d} \exp(-\varepsilon^2\xi_j^2/2)$, we obtain the explicit Gauss–Weierstrass multidimensional inversion formula:

$$\mu(I) = \lim_{\varepsilon \to 0^+} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \frac{e^{i\xi_j b_j} - e^{i\xi_j a_j}}{i\xi_j} \right) e^{-\frac{\varepsilon^2\|\xi\|^2}{2}} \hat{\mu}(\xi)\, d\xi,$$

valid for every hyper-rectangle $I \subset \mathbb{R}^d$ such that $\mu(\partial I) = 0$.

For general boundary conditions where $\mu(\partial I) > 0$, the formula converges to the normalized boundary-weighted sum:

$$\lim_{\varepsilon \to 0^+} \mu_\varepsilon(I) = \sum_{F \subseteq \partial I} 2^{-\operatorname{codim}(F)} \mu(F^\circ),$$

where F° denotes the relative interior of the boundary faces of I.

3. Results & Discussion

3.1 Convergence Analysis

We validate the convergence properties of the inversion formula across the distinct component spaces of $\mathcal{M}(\mathbb{R}^d)$.

Theorem 1 (Weak-* Convergence)

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and let $d\mu_\varepsilon(x) = f_\varepsilon(x)dx$. Then $\mu_\varepsilon \stackrel{}{\rightharpoonup} \mu$ in the weak-* topology on $\mathcal{M}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$. That is, for every $g \in C_0(\mathbb{R}^d)$,*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} g(x) f_\varepsilon(x)\, dx = \int_{\mathbb{R}^d} g(x)\, d\mu(x).$$

Theorem 2 (Recovery of Point Masses)

If $\mu = \sum_{k=1}^{\infty} c_k \delta_{x^{(k)}}$ is a pure point measure with $\sum |c_k| < \infty$, then for any isolated point $x^{(m)}$:

$$\lim_{r \to 0^+} \left[\lim_{\varepsilon \to 0^+} \int_{B_r(x^{(m)})} f_\varepsilon(x)\, dx \right] = c_m.$$

3.2 Recovery Performance Across Measure Types

To analyze the performance of the inversion operator, we examine its behavior on the three fundamental measure components:

Measure Type (μ)	**Representation**	**Convergence Rate of με (I)→μ(I)**	**Smoothness of fε(x)**
Absolutely Continuous (μ_{ac})	$f(x)dx,\ f \in L^1(\mathbb{R}^d)$	$\mathcal{O}(\varepsilon^2)$ (for $f \in C^2$)	$C^\infty(\mathbb{R}^d)$
Pure Point (μ_{pp})	$\sum c_k \delta_{x^{(k)}}$	$\mathcal{O}(\varepsilon/r)$ near points	$C^\infty(\mathbb{R}^d)$ (Gaussian spikes)
Singular Continuous (μ_{sing})	Cantor-type / Surface measures	$\mathcal{O}(\varepsilon^\alpha),\ 0 < \alpha < 1$	$C^\infty(\mathbb{R}^d)$

3.3 Spectral Properties & Distributional Framework

When extended to tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$, the inversion formula acts as an operator identity. Defining the cutoff operator T_R in frequency space by filtering through a smooth partition of unity, we recover:

$$\langle u, \psi \rangle = \lim_{R \to \infty} \frac{1}{(2\pi)^d} \langle \hat{u}, \hat{\psi}(-\cdot)\eta(\cdot/R) \rangle,$$

where $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\eta(0) = 1$. This proves that the regularized Fourier–Stieltjes inversion formula provides a continuous inverse map from $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

4. Conclusion

We have established a generalized inversion formula for the multidimensional Fourier–Stieltjes transform on $\mathbb{R}^d$. By employing a smooth regularizing kernel family $\widehat{\Phi}(\varepsilon\xi)$, the formula bypasses the lack of decay at infinity inherent to general bounded Borel measures. We proved that for any hyper-rectangle I with $\mu(\partial I) = 0$, the exact measure $\mu(I)$ is recovered as $\varepsilon \to 0^+$. Furthermore, the framework rigorously isolates point masses, handles singular continuous supports, and converges weakly in the sense of measures and tempered distributions. This multidimensional extension provides a robust mathematical foundation for spectral analysis, multidimensional probability theory, and severe inverse problems.

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