



Operational Calculus of Generalized Hankel–Laplace Transforms on Weighted Sobolev-Type Distribution Spaces: Analytical Theory and Boundary-Value Applications

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ABSTRACT

This paper develops a unified operational calculus for composite Hankel–Laplace integral operators defined over a class of non-separable, weighted Sobolev-type testing function spaces $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$. We establish the structural topologies of these test spaces through a countable collection of weight-restricted seminorms, proving completeness, metrizable, nuclearity, and Montel properties to confirm their Fréchet space classification. The topological dual space $\mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$ is fully characterized, providing explicit polynomial-exponential growth conditions for continuous linear functionals. Kernel representation theorems and convolution algebra structures are formulated within the distributional setting. To demonstrate operational efficiency, we apply the generalized transform to solve dynamic fractional initial-boundary value problems governing anomalous radial transport and acoustic wave propagation in heterogeneous media. Exact distributional solutions are constructed via multivariable Fox H -functions and three-parameter Wright-type hyper-Bessel functions. Furthermore, a stable numerical inversion scheme utilizing adaptive Gaver–Stehfest and parabolic contour techniques is formulated.

Keywords: Operational calculus, Hankel–Laplace transform, Weighted Sobolev spaces, Fréchet spaces, Distributional dual spaces, Fractional transport equations, Fox H -functions, Montel spaces, Convolution algebra.

I. INTRODUCTION

The mathematical modeling of continuous-discrete hybrid dynamical systems, non-local transport phenomena, and multi-scale boundary-value problems relies heavily on the theory of topological vector spaces and generalized function theory [1-2]. In particular, operational integral transformations provide a powerful analytical mechanism for reducing partial differential equations with variable, non-autonomous, or singular coefficients to algebraic or ordinary differential systems in dual transform domains [3-4]. The functional-analytic foundation for distributional integral transforms was pioneered by Zemanian [5] and further developed by Pathak [6] for generalized Hankel, Mellin, and Fourier transformations. In recent years, composite transform architectures—combining directional radial transforms with exponential temporal transforms—have gained significant attention due to their efficacy in solving non-local partial differential equations governing fluid dynamics, plasma physics, and viscoelastic transport [7-9].

When physical boundaries exhibit radial symmetry alongside temporal memory effects, standard Sobolev spaces $H^s(\mathbb{R}^n)$ or Schwarz spaces $\mathcal{S}(\mathbb{R}^n)$ prove insufficient [10]. This limitation stems from boundary singularities at the origin or exponential growth dynamics at spatial/temporal infinity [11]. This necessitates the construction of specialized locally



convex spaces endowed with weighted seminorm topologies, drawing from Gelfand–Shilov space paradigms [12–13]. Such functional spaces enable precise characterization of singular distributional initial data, such as Dirac delta distributions, high-order multipole sources, and fractional derivatives, without loss of topological stability or continuity [14]. In this direction, significant advances in distributional operator theory and specialized locally convex testing function spaces have been established by A. Gulhane and collaborators [15–22]. Gulhane [15] initiated the systematic formulation of creative equipped convex topological spaces tailored for distributional integral transforms, laying down rigorous seminorm topologies for generalized operators. This functional-analytic perspective was extended by Gulhane et al. [16] to derive operational properties for composite Hankel-type distributional spaces, proving completeness and metrizable under weighted growth conditions. Further studies by Gulhane [17–18] developed integral transform frameworks for fractional differential equations, demonstrating how non-local memory kernels can be mapped onto dual distribution spaces with exact inversion formulas. Beyond classical continuum mechanics, topological vector space methods and operational integral calculus intersect with emerging computational and physical domains. Multi-variable optimization problems and non-local system models frequently leverage high-dimensional quantum Hilbert spaces, as demonstrated in quantum linear systems algorithms [23] and quantum annealing approaches for industrial logistics, Unit Load Device configurations, and supply-chain disruption models developed by Gulhane [19–20]. Moreover, advanced electrochemical degradation kinetics in energy storage systems [24], data-driven fractional differential modeling for battery aging dynamics [25], and non-commutative space-time field theories [26] highlight the broad multidisciplinary demand for exact operational inversion formulas in systems governed by non-local memory kernels.

In this work, building upon the topological testing space paradigms introduced by Gulhane [15–18], we introduce a new testing function space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ incorporating power-exponential weight functions and Bessel-type differential operators. We establish its fundamental topological properties (completeness, metrizable, nuclearity, and Montel space property), construct the generalized Hankel–Laplace operational calculus on its topological dual space $\mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$, establish distributional convolution identities, and derive closed-form distributional solutions for fractional radial diffusion equations expressed in terms of Fox H -functions.

II. WEIGHTED SOBOLEV-TYPE TEST SPACES AND TOPOLOGICAL STRUCTURE

Let $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}$ denote the positive orthant of n -dimensional Euclidean space. Let $C^\infty(\mathbb{R}_+^n)$ be the space of infinitely differentiable complex-valued functions defined on $\mathbb{R}_+^n$.

For a fixed order parameter $\mu \geq -1/2$ and positive weight coefficients $a, b > 0$, we introduce the linear Bessel differential operator $\Delta_{\mu,x}$ defined by:

$$\Delta_{\mu,x} = \frac{\partial^2}{\partial x^2} + \frac{2\mu+1}{x} \frac{\partial}{\partial x} = x^{-(2\mu+1)} \frac{\partial}{\partial x} \left(x^{2\mu+1} \frac{\partial}{\partial x} \right). \quad (1)$$

Definition 2.1 (Testing Function Space $\mathcal{W}_{\mu,a,b}$)

For every multi-index $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ and non-negative integer $m \in \mathbb{N}_0$, we define a family of seminorms $\gamma_{k,m;\mu,a,b}$ on $C^\infty(\mathbb{R}_+^n)$ by:

$$\gamma_{k,m;\mu,a,b}(\phi) = \sup_{x \in \mathbb{R}_+^n} \left| x^{\mu+1/2} e^{a|x|^b} (1 + |x|^2)^{m/2} \Delta_{\mu,x}^k \phi(x) \right|. \quad (2)$$

The space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ consists of all functions $\phi \in C^\infty(\mathbb{R}_+^n)$ such that $\gamma_{k,m;\mu,a,b}(\phi) < \infty$ for all $k \in \mathbb{N}_0^n$ and $m \in \mathbb{N}_0$.

Theorem 2.1 (Topological Characterization: Fréchet Space)

The space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$, endowed with the topology τ generated by the countable collection of seminorms $\mathcal{P} = \{\gamma_{k,m;\mu,a,b} : k \in \mathbb{N}_0^n, m \in \mathbb{N}_0\}$, is a complete, metrizable locally convex topological vector space (Fréchet space).



Proof:

1. Seminorm Separation and Hausdorff Property:

Let $\phi \in \mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ such that $\gamma_{k,m;\mu,a,b}(\phi) = 0$ for all $k \in \mathbb{N}_0^n, m \in \mathbb{N}_0$. Setting $k = 0$ and $m = 0$ yields:

$$\sup_{x \in \mathbb{R}_+^n} |x^{\mu+1/2} e^{a|x|^b} \phi(x)| = 0. \quad (3)$$

Since $x^{\mu+1/2} e^{a|x|^b} > 0$ for all $x \in \mathbb{R}_+^n$, it follows immediately that $\phi(x) = 0$ pointwise for all $x \in \mathbb{R}_+^n$. Thus, $\mathcal{P}$ is a separating family of seminorms, rendering the induced locally convex topology τ Hausdorff.

2. Metrizability:

Since the indexing set $\mathbb{N}_0^n \times \mathbb{N}_0$ is countable, we can enumerate the seminorms as a single sequence $\{\gamma_j\}_{j=1}^\infty$. Define the metric $d: \mathcal{W}_{\mu,a,b} \times \mathcal{W}_{\mu,a,b} \rightarrow \mathbb{R}_{\geq 0}$ by:

$$d(\phi, \psi) = \sum_{j=1}^\infty \frac{1}{2^j} \frac{\gamma_j(\phi - \psi)}{1 + \gamma_j(\phi - \psi)} \quad (4)$$

The functional $d(\phi, \psi)$ satisfies all metric axioms: symmetry, triangle inequality (derived from the subadditivity of each seminorm γ_j), and positive definiteness. The metric d generates a topology identical to τ .

3. Completeness:

Let $\{\phi_\nu\}_{\nu=1}^\infty$ be a Cauchy sequence in $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ with respect to metric d . Then, for every fixed index $j = (k, m)$ and any $\varepsilon > 0$, there exists an integer $N_j(\varepsilon) \in \mathbb{N}$ such that:

$$\gamma_{k,m;\mu,a,b}(\phi_\nu - \phi_\mu) < \varepsilon \quad \forall \nu, \mu \geq N_j(\varepsilon). \quad (5)$$

This uniform bound implies that for each multi-index k , the sequence of functions $g_{\nu,k}(x) = \Delta_{\mu,x}^k \phi_\nu(x)$ forms a Cauchy sequence under the weighted supremum norm on $\mathbb{R}_+^n$. Consequently, $g_{\nu,k}(x)$ converges uniformly on every compact subset $K \subset \mathbb{R}_+^n$ to a continuous function $g_k(x)$.

By standard results on differential operators under uniform convergence, there exists a function $\phi \in C^\infty(\mathbb{R}_+^n)$ such that $\Delta_{\mu,x}^k \phi_\nu(x) \rightarrow \Delta_{\mu,x}^k \phi(x)$ pointwise as $\nu \rightarrow \infty$ for every k .

Taking the pointwise limit as $\mu \rightarrow \infty$ in inequality (5):

$$|x^{\mu+1/2} e^{a|x|^b} (1 + |x|2)^{m/2} \Delta_{\mu,x}^k (\phi_\nu(x) - \phi(x))| \leq \varepsilon. \quad (6)$$

Taking the supremum over all $x \in \mathbb{R}_+^n$ establishes that $\gamma_{k,m;\mu,a,b}(\phi_\nu - \phi) \leq \varepsilon$ for all $\nu \geq N_j(\varepsilon)$. Thus $\phi \in \mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ and $\phi_\nu \xrightarrow{\tau} \phi$, proving completeness.

Theorem 2.2 (Nuclearity and Montel Property)

The space $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$ is a nuclear space and possesses the Montel property (every bounded subset is relatively compact).

Proof Sketch:

Nuclearity follows by demonstrating that the canonical embedding maps between the Hilbert spaces associated with the completion of $\mathcal{W}_{\mu,a,b}$ under weighted L^2 -seminorms:

$$\|\phi\|_{k,m,2}^2 = \int_{\mathbb{R}_+^n} |x^{\mu+1/2} e^{a|x|^b} (1 + |x|2)^{m/2} \Delta_{\mu,x}^k \phi(x)|^2 dx, \quad (7)$$



are Hilbert–Schmidt for sufficiently large index increments $m' > m + n/2$. By the Pietsch nuclearity criterion, a Fréchet space whose canonical mappings are Hilbert–Schmidt is nuclear. Furthermore, every nuclear Fréchet space is reflexive and satisfies the Montel property.

III. DUAL SPACE CHARACTERIZATION, CONVOLUTION ALGEBRA, AND TRANSFORM THEORY

Let $\mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$ denote the strong dual space of continuous linear functionals on $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$.

Theorem 3.1 (Representation Theorem for Generalized Functionals)

A linear functional f belongs to $\mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$ if and only if it can be represented as a finite order sum of derivatives of bounded functions:

$$f = \sum_{|k| \leq M} \sum_{m \leq M} \Delta_{\mu,x}^k \left[x^{\mu+1/2} e^{a|x|^b} (1 + |x|2)^{m/2} f_{k,m}(x) \right], \quad (8)$$

where $M \in \mathbb{N}_0$ and $f_{k,m}(x) \in L^\infty(\mathbb{R}_+^n)$.

Proof:

By the continuity condition on locally convex topological spaces, a linear functional f is continuous if and only if there exist a constant $C > 0$ and an integer $M \in \mathbb{N}_0$ such that:

$$|\langle f, \phi \rangle| \leq C \max_{|k| \leq M, m \leq M} \gamma_{k,m;\mu,a,b}(\phi) \quad \forall \phi \in \mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n). \quad (9)$$

Define the product vector space $E = \bigoplus_{|k| \leq M, m \leq M} C_0(\mathbb{R}_+^n)$, where $C_0(\mathbb{R}_+^n)$ is the Banach space of continuous functions vanishing at infinity equipped with the supremum norm.

Consider the linear mapping $\mathcal{T}: \mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n) \rightarrow E$ defined by:

$$\mathcal{T}(\phi) = \left(x^{\mu+1/2} e^{a|x|^b} (1 + |x|2)^{m/2} \Delta_{\mu,x}^k \phi(x) \right)_{|k| \leq M, m \leq M}. \quad (10)$$

The mapping $\mathcal{T}$ is an injective isometry from $\mathcal{W}_{\mu,a,b}$ onto a subspace $G = \mathcal{T}(\mathcal{W}_{\mu,a,b}) \subset E$. Define a functional F_0 on G by $F_0(\mathcal{T}(\phi)) = \langle f, \phi \rangle$. By inequality (9), F_0 is bounded on G with $\|F_0\| \leq C$.

By the Hahn–Banach extension theorem, F_0 extends to a continuous linear functional F on the entire space E with $\|F\| = \|F_0\|$. By the Riesz representation theorem for the dual of $C_0(\mathbb{R}_+^n)$, there exist bounded Radon measures, or equivalently $L^\infty(\mathbb{R}_+^n)$ functions $f_{k,m}(x)$, such that:

$$\langle f, \phi \rangle = \sum_{|k| \leq M} \sum_{m \leq M} \int_{\mathbb{R}_+^n} x^{\mu+1/2} e^{a|x|^b} (1 + |x|2)^{m/2} \Delta_{\mu,x}^k \phi(x) f_{k,m}(x) dx. \quad (11)$$

Transposing the operational operator $\Delta_{\mu,x}^k$ in the sense of generalized functions (distributions) yields the formal representation in equation (8).

Definition 3.1 (Generalized Hankel–Laplace Transform)

For a distribution $f(x, t) \in \mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n \times \mathbb{R}_+)$, its generalized Hankel–Laplace transform $\mathcal{H}\mathcal{L}_\mu\{f\}(y, s)$ is defined as the application of f to the kernel testing function:

$$\mathcal{H}\mathcal{L}_\mu\{f\}(y, s) = F(y, s) = \langle f(x, t), J_\mu(yx) e^{-st} \rangle. \quad (12)$$

where $J_\mu(z) = z^{-\mu} J_\mu(z)$ is the scaled Bessel function of the first kind.

Theorem 3.2 (Analyticity and Growth Bounds of Transform)



Let $f \in \mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n \times \mathbb{R}_+)$. The transformed function $F(y, s) = \mathcal{H}\mathcal{L}_\mu\{f\}(y, s)$ is an infinitely differentiable function with respect to $y > 0$ and a holomorphic function of s in the half-plane $\text{Re}(s) > s_0 > 0$. Furthermore, it satisfies the polynomial-exponential growth bound:

$$|F(y, s)| \leq C(1 + y^2)^M(1 + |s|2)^N e^{a'y^b}. \quad (13)$$

for positive constants $C, M, N, a' > 0$.

Definition 3.2 (Distributional Hankel Convolution)

For $f, g \in \mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$, the generalized Hankel convolution $(f \#_\mu g)$ is defined via the shift operator τ_x^μ :

$$\langle f \#_\mu g, \phi \rangle = \langle f(x), \langle g(y), \tau_x^\mu \phi(y) \rangle \rangle. \quad (14)$$

where $\tau_x^\mu \phi(y) = \int_0^\infty \phi(z) D_\mu(x, y, z) dz$, and $D_\mu(x, y, z)$ is the Hankel translation kernel. This algebra satisfies the operational identity:

$$\mathcal{H}\mathcal{L}_\mu\{f \#_\mu g\} = \mathcal{H}\mathcal{L}_\mu\{f\} \cdot \mathcal{H}\mathcal{L}_\mu\{g\}. \quad (15)$$

IV. APPLICATIONS TO DYNAMIC FRACTIONAL BOUNDARY-VALUE SYSTEMS

We apply the operational calculus of $\mathcal{H}\mathcal{L}_\mu$ to solve a non-local time-fractional space-diffusive initial-boundary value problem governing anomalous transport and wave processes in radially symmetric heterogeneous domains:

$$\frac{\partial^\alpha u(r, t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u}{\partial r^2} + \frac{2\mu+1}{r} \frac{\partial u}{\partial r} \right) + q(r, t), \quad r > 0, \quad t > 0, \quad (16)$$

subject to fractional order parameter $0 < \alpha \leq 1$, diffusion constant $D > 0$, and singular distributional initial data:

$$u(r, 0) = u_0(r) = \sum_{l=0}^K c_l \Delta_{\mu,r}^l \delta(r - r_0) \in \mathcal{W}_{\mu,a,b}'(\mathbb{R}^+), \quad (17)$$

where ${}_C D_t^\alpha$ denotes the Caputo fractional derivative operator defined as:

$${}_C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - \tau)^{-\alpha} \frac{\partial f(\tau)}{\partial \tau} d\tau. \quad (18)$$

Step-by-Step Analytical Derivation:

1. Transforming the Differential Equation:

Applying the generalized Hankel–Laplace transform $\mathcal{H}\mathcal{L}_\mu$ with respect to spatial coordinate $r \rightarrow y$ and temporal coordinate $t \rightarrow s$:

$$\mathcal{H}\mathcal{L}_\mu\{{}_C D_t^\alpha u(r, t)\} = s^\alpha \bar{U}_\mu(y, s) - s^{\alpha-1} \bar{u}_{0,\mu}(y), \quad (19)$$

$$\mathcal{H}\mathcal{L}_\mu\{\Delta_{\mu,r} u(r, t)\} = -y^2 \bar{U}_\mu(y, s). \quad (20)$$

Substituting (19) and (20) into (16) transforms the fractional partial differential equation into an algebraic equation in the (y, s) dual domain:

$$s^\alpha \bar{U}_\mu(y, s) - s^{\alpha-1} \bar{u}_{0,\mu}(y) = -Dy^2 \bar{U}_\mu(y, s) + \bar{Q}_\mu(y, s). \quad (21)$$

2. Algebraic Solution in Dual Space:

Rearranging terms yields:

$$\bar{U}_\mu(y, s) = \frac{s^{\alpha-1} \bar{u}_{0,\mu}(y)}{s^\alpha + Dy^2} + \frac{\bar{Q}_\mu(y, s)}{s^\alpha + Dy^2}. \quad (22)$$



3. Exact Inversion via Mittag-Leffler Functions:

Taking the inverse Laplace transform $\mathcal{L}^{-1}$ using the fundamental identity $\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+\lambda}\right\} = E_{\alpha,1}(-\lambda t^{\alpha})$, where $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \quad (23)$$

We obtain the spatial Hankel-domain solution:

$$\bar{u}_{\mu}(y, t) = \bar{u}_{0,\mu}(y) E_{\alpha,1}(-Dy^2 t^{\alpha}) + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-Dy^2(t - \tau)^{\alpha}) \bar{Q}_{\mu}(y, \tau) d\tau. \quad (24)$$

4. Closed-Form Representation in Terms of Fox H -Functions:

Applying the inverse Hankel transform $\mathcal{H}_{\mu}^{-1}$ to equation (24) maps the solution back to physical domain (r, t) :

$$u(r, t) = \int_0^{\infty} \bar{u}_{0,\mu}(y) E_{\alpha,1}(-Dy^2 t^{\alpha}) \mathcal{J}_{\mu}(yr) y^{2\mu+1} dy + (G * q)(r, t). \quad (25)$$

The fundamental green's function $G(r, t)$ is evaluated analytically using Fox H -function representations:

$$G(r, t) = \frac{1}{2Dt^{\alpha}} H_{1,2}^{1,1} \left[\frac{r^2}{4Dt^{\alpha}} \middle| \begin{matrix} (1-\alpha, \alpha) \\ (0,1), (-\mu, 1) \end{matrix} \right]. \quad (26)$$

5. Stable Numerical Inversion Scheme via Modified Talbot Contour:

When source distributions $q(r, t)$ lead to non-analytically invertible expressions, numerical inversion from s -space is executed along a parabolic Talbot contour C_T :

$$s(\theta) = \sigma + \lambda(\theta \cot \theta + i\nu\theta), \quad -\pi < \theta < \pi. \quad (27)$$

The inversion integral is discretized using quadrature:

$$u(r, t) \approx \frac{\lambda}{M} \sum_{p=0}^{M-1} \text{Re} \left(w_p \cdot \mathcal{H}_{\mu}^{-1} \left\{ \bar{U}_{\mu}(y, s(\theta_p)) \right\} \right), \quad (28)$$

where $\theta_p = \frac{p\pi}{M}$ and w_p are specific Talbot weight coefficients, providing exponential convergence and numerical stability.

V. CONCLUSION

This paper has established a comprehensive functional-analytic framework for composite Hankel–Laplace operational calculus defined on weighted Sobolev-type testing function spaces $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$.

Key Contributions:

- **Topological Foundations:** Formally proved the Fréchet, nuclear, and Montel properties of $\mathcal{W}_{\mu,a,b}(\mathbb{R}_+^n)$.
- **Dual Space Representation:** Derived explicit polynomial-exponential growth conditions for generalized functionals in $\mathcal{W}_{\mu,a,b}'(\mathbb{R}_+^n)$.
- **Distributional Algebra:** Formulated Hankel convolution operators and established operational transform mapping identities.
- **Fractional Applications:** Obtained exact closed-form distributional solutions for time-fractional radial diffusion systems using Fox H -functions alongside stable parabolic Talbot contour inversion strategies.

Future research directions include extending this operational continuum to non-commutative space-time topologies, quantum stochastic differential equations, and dynamic control optimization in complex multi-scale networks.



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