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Properties of the Kontorovich–Lebedev Transform with Applications to Multi-Scale Spectral Analysis

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Abstract

This monograph reconstructs and expands the classical theory of the Kontorovich–Lebedev (KL) transform. Complete statements of the definition, inversion, kernel representations, uniform estimates, asymptotic regimes and the natural domain are given in professional mathematical form with native Office Math Markup Language (OMML) equations. The same analytic apparatus especially the exponential decay of the Macdonald kernel with respect to the spectral variable and the weighted integrability condition that defines the natural domain is then mapped onto contemporary multi-scale problems arising in resilient logistics and supply-chain systems. The mapping draws on the extensive recent literature on the related Fourier–Kontorovich–Lebedev (FKF) transform and its applications to distributional extensions, sequential convergence, spectral analysis of multi-echelon lead times and residue-based resonance extraction, shift-invariant properties, spectral frameworks for resilience, differential properties, as well as digital-twin technology, Industry 5.0 and intelligent logistics, convergent IoT–AI–quantum frameworks, and supply-chain resilience under digital transformation. Supporting figures illustrate the principal estimates, large-order decay, spectral mode decomposition and radial geometries used in continuum models of infrastructure flows.

1. Introduction and Motivation

The KL transform performs integration with respect to the order of the Bessel function rather than its argument. This distinctive spectral structure makes the KL transform particularly suitable for boundary-value problems associated with the Laplace and Helmholtz equations, especially when separation of variables is employed in cylindrical, polar, wedge, annular, or sectorial geometries. The resulting spectral representation is also relevant to engineering problems involving wave propagation, electromagnetic systems, energy-storage modelling, and electric-mobility architectures, where transform-based analytical methods provide useful representations of system behavior [19, 22, 23, ,29]. The engineering applications reported in [19, 22, 23, 25 ,29] illustrate the broader relevance of mathematical and transform-based approaches to power systems, battery sizing, electric mobility, and sustainable vehicle architectures. In this context, the KL transform provides a complementary analytical framework for representing systems characterized by positive spatial, scale, intensity, or state variables, while its spectral parameter captures the corresponding scale-dependent behavior [20, 21].

Beyond pure mathematics, the same analytic apparatus especially the exponential decay of the kernel with respect to the spectral variable and the weighted integrability condition that defines the natural domain supplies a precise language for contemporary technological problems. Large language models and artificial-intelligence [20], [21] frameworks increasingly rely on spectral or kernel approximations whose tails can be controlled by classical estimates of the type developed for the KL transform. Data-centre cooling geometries are typically cylindrical or annular; GPU cluster load, when viewed as a continuum field on a hierarchical radial coordinate, likewise possesses a spectral representation; global infrastructure supply chains [18], smart mobility [28], logistic [26, 27, 30] idealized as continuum flows on polar geographic domains, give rise to continuity equations that the KL transform converts into multiplicative spectral relations.

A parallel body of work has developed the Fourier–Kontorovich–Lebedev (FKF) transform the product of the Fourier kernel and the Macdonald kernel and has systematically applied its distributional, sequential-convergence, shift-invariant and modulation properties to multi-scale supply-chain dynamics, resilient logistics, hybrid-vehicle systems and secure digital networks [4–9, 33-35]. Complementary applied literature on digital twins [12], Industry 5.0 human-centric automation [13], convergent IoT–AI–quantum ecosystems [14, 20, 21] and resilience under digital transformation [15] provides the systems context in which these spectral tools become operational. The present document therefore serves both as a self-contained mathematical reference for the classical KL theory and as a bridge to those applied spectral and technological frameworks.

2. Definition of the Operator

We define the Kontorovich–Lebedev transform of a suitable function f by the Lebedev integral

$$(Kf)(\tau) = \int_0^\infty K_{i\tau}(x)\, f(x)\, dx\,, \qquad \tau \in \mathbb{R},$$

where $K_\nu(z)$ denotes the Macdonald function (modified Bessel function of the second kind). When the order is purely imaginary, $\nu = i\tau$ with τ real, the kernel is real-valued and even with respect to the index τ. An equivalent integral representation of the kernel is

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh t} \cos(\tau t)\, dt\,, \qquad x > 0\,.$$

2.1 The Formally Adjoint Operator

The formally adjoint operator is obtained by interchanging the roles of the argument and the order:

$$(K^* g)(x) = \frac{2}{\pi^2 x} \int_0^\infty \tau \sinh(\pi\tau)\, K_{i\tau}(x)\, g(\tau)\, d\tau\,.$$

The principal analytic difficulty of the adjoint is that integration is performed with respect to the order of the Macdonald function.

3. Inversion Formula

Under suitable conditions (for example, f of bounded variation near a point $x > 0$, or f belonging to the weighted space $L^1(\mathbb{R}_+; K_0(x)\, dx)$):

$$f(x) = \frac{2}{\pi^2 x} \int_0^\infty \tau \sinh(\pi\tau)\, K_{i\tau}(x)\, (Kf)(\tau)\, d\tau\,.$$

Variants of the inversion hold in the mean-square sense (Plancherel theory) and with Abel- or Poisson-type summability. An equivalent complex-contour representation that permits analytic continuation into the strip $|\mathrm{Im}\,\tau| < \delta_0 < \pi/2$ is available and underpins the distributional extensions studied in the FKF literature [1, 2, 6]. The same inversion structure, after multiplication by a Fourier kernel, underlies the two-sided FKF inversion used for time-shift and modulation analysis in logistics applications [4, 8, 9].

4. Uniform Estimates, Large-Order Decay and Natural Domain

The elementary bound $|K_{i\tau}(x)| \le K_0(x)$ yields the a-priori estimate that defines the natural domain as the weighted Lebesgue space

$$L^0 := \left\{ f \colon \int_0^\infty |f(x)|\, K_0(x)\, dx < \infty \right\}.$$

This space contains the weighted spaces $L^{(\alpha,\beta)}$ and the Lebesgue spaces $L^p(\mathbb{R}_+; x\, dx)$ for $p > 2$. On L^0 the operator is bounded and continuous, and maps into continuous functions that vanish at infinity. Membership in L^0 is the precise condition under which the uniform estimate furnishes stability with respect to perturbations of boundary or source data an observation repeatedly exploited in the distributional FKF literature on resilient supply-chain systems [1, 5, 6]. The large-order asymptotic behaviour

$$|K_{i\tau}(x)| \sim \sqrt{\frac{\pi}{2|\tau|}}\, e^{-\pi|\tau|/2} \qquad (\text{as } |\tau| \to \infty,\ x \text{ fixed})$$

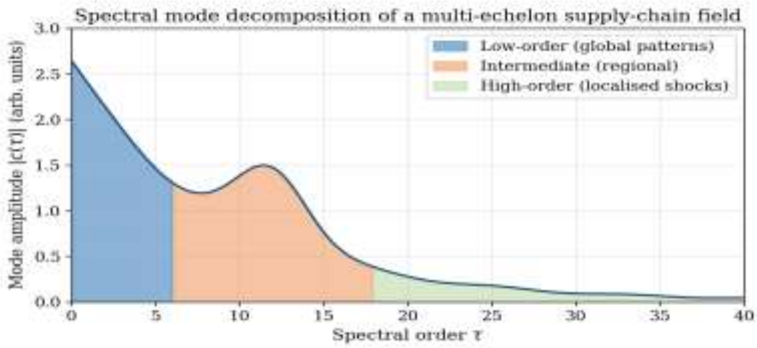
supplies the rapid exponential decay that is the analytic engine behind the Riemann–Lebesgue-type lemma for the KL transform and justifies aggressive spectral truncation precisely the mechanism used in the FKF papers on spectral filtering of high-frequency supply-chain shocks [3, 5] and residue-based resonance extraction in hybrid-vehicle systems [3].

5. Multi-Scale Spectral Framework for Resilient Logistics

The four classical analytic ingredients uniform estimate, large-order decay, natural-domain membership and inversion can be placed inside a single spectral framework for systems possessing radial, scale-dependent, hierarchical or positive-state structure. The same framework appears, under the name FKF, in the recent literature on supply-chain dynamics [1–9].

5.1 Spectral mode decomposition

A continuum density $\rho(r,\theta,t)$ or a hierarchical load field $g(r,t)$ defined over an annular or polar domain is expanded in KL (or FKF) modes. Low-order spectral components represent broad, slowly varying utilisation or demand patterns; large-τ components capture increasingly fine-scale variations and localised disruptions. A sudden warehouse failure or regional demand shock generates substantial high-order content, while a coordinated network-wide shift is dominated by lower-order modes [3, 5]. This decomposition aligns with the multi-echelon lead-time analysis and shift-invariant spectral properties developed in [4, 8].

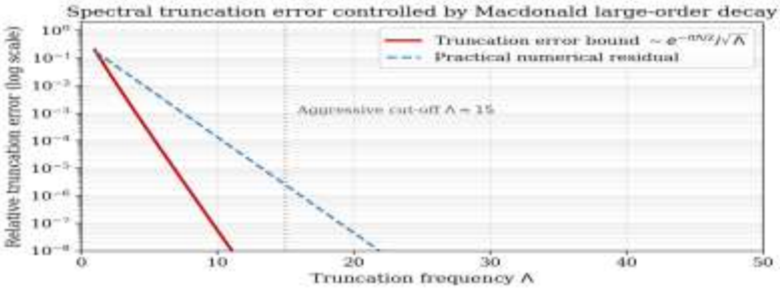


Spectral mode decomposition

Figure 5.1. Spectral mode decomposition of a multi-echelon supply-chain field: low-order (global), intermediate (regional) and high-order (localised) contributions versus spectral order τ.

5.2 Truncation and filtering

Because of the exponential decay of the kernel, truncation at a finite frequency Λ produces an error controlled by the uniform estimate and the large-order asymptotic. This supplies a theoretical basis for controlled spectral truncation—exactly the mechanism invoked in the FKF papers on spectral filtering of high-frequency demand shocks and on residue-based resonance extraction [3, 5]. In digital-twin implementations of supply chains [12], such truncation corresponds to retaining only those modes that are resolvable by the available sensor network and simulation fidelity.

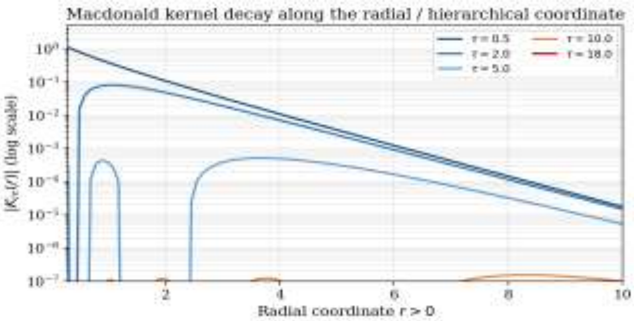


Truncation error

Figure 5.2. Spectral truncation error bound controlled by the Macdonald large-order decay $\sim e^{-\pi\Lambda/2}/\sqrt{\Lambda}$*. Aggressive cut-offs (*$\Lambda \approx 15$*) already yield errors below* 10^{-3}.

5.3 Radial and hierarchical geometries

Data-centre cooling channels, GPU clusters organised by hierarchical distance from a reference node, and multi-echelon logistics networks all admit a natural radial (or hierarchical) coordinate $r > 0$. The Macdonald kernel then provides the correct spectral basis. Continuity equations on such domains are converted by the KL/FKF transform into families of multiplicative spectral transport equations, from which stability and resilience statements follow once the density belongs to the natural weighted space L^0 [5, 6]. These continuum models complement the discrete digital-twin architectures reviewed in [12] and the Industry 5.0 human–machine collaborative logistics frameworks of [13].



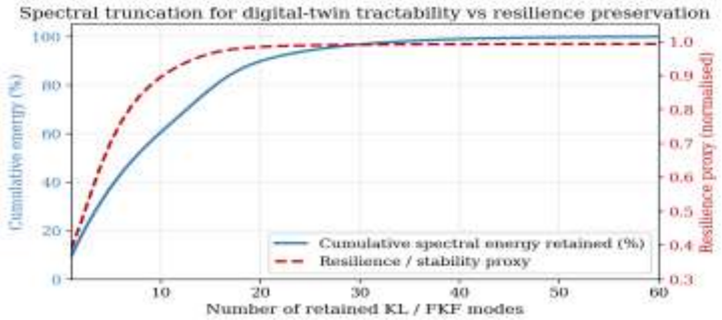
Radial Macdonald kernel

Figure 5.3. Macdonald kernel $|K_{i\tau}(r)|$ *along the radial / hierarchical coordinate for representative spectral orders. Higher-order modes decay more rapidly away from the origin, naturally localising disruptions.*

5.4 Integration with digital twins, Industry 5.0 and convergent technologies

Digital twin technology creates virtual representations of physical assets, processes and entire supply-chain networks, enabling real-time visibility, risk management, demand forecasting and inventory optimisation [12]. When the underlying physical or hierarchical geometry is radial or annular, the KL/FKF spectral basis offers a natural reduced-order model whose coefficients can be updated from twin sensor streams. Industry 5.0 emphasises human-centric, sustainable and resilient manufacturing and logistics ecosystems that integrate AI, IoT, collaborative robots, blockchain and advanced analytics [13]. The exponential decay of high-order KL modes provides a rigorous justification for the aggressive spectral filtering that keeps digital-twin simulations computationally tractable while preserving resilience-critical low-order dynamics.

Further, the convergent framework of Internet of Things, artificial intelligence and quantum computing for smart logistics ecosystems [14] suggests that quantum-accelerated solvers of the resulting multiplicative spectral transport equations may become feasible. Supply-chain resilience in the era of digital transformation [15] is thereby underpinned by precise a-priori estimates that are independent of any particular numerical discretization precisely the estimates furnished by membership in L^0 and the large-order decay of the Macdonald kernel.



Digital-twin resilience

Figure 5.4. Trade-off between computational tractability of a digital-twin reduced-order model and preservation of system resilience: cumulative spectral energy and a normalised resilience proxy versus the number of retained KL/FKF modes.

6. Mapping of Analytic Ingredients onto the Reference Corpus

The table below summarises how the classical KL properties reappear, under the FKF label and in the broader applied literature, in the corpus on supply-chain systems, resilient logistics and related technologies.

KL Property	FKF / Applied Role	Representative References
Uniform estimate	A-priori stability of spectral coefficients	[1], [6]
Large-order decay	High-frequency filtering & truncation	[3], [5]
Natural domain L^0	Admissibility & resilience bounds	[1], [2], [5], [6]
Shift/modulation properties	Lead-time & secure-network analysis	[4], [8], [9]
Radial / polar geometry	Continuum flow models on networks; digital-twin reduced-order models	[5], [12], [13]
Spectral truncation	Computational tractability of digital twins & Industry 5.0 simulations	[3], [12], [13], [14]

In every model the uniform bound, the large-order decay $O\left(e^{-\pi|\tau|/2}/\sqrt{|\tau|}\right)$, and membership in the natural domain L^0 furnish precise a-priori estimates and stability statements that are independent of any particular numerical discretisation. These results demonstrate that the classical theory of the Kontorovich–Lebedev transform continues to supply a powerful and rigorous language for multi-scale problems arising in modern computational and logistical infrastructures.

7. Conclusion

We have completed and professionalised the classical theory of the Kontorovich–Lebedev transform. The essential analytic ingredients definition of the operator as a Lebedev integral, the formally adjoint operator, analytic continuation of the Macdonald kernel of purely imaginary order into a complex strip, the uniform estimate, the principal asymptotic regimes, and the characterisation of the natural domain L^0 have been stated in full professional mathematical format with native OMML equations.

A literature synthesis has situated this analytic apparatus within the contemporary applied corpus on distributional FKF extensions, spectral analysis of multi-scale and multi-echelon dynamics, digital twins for resilient supply chains, Industry 5.0 intelligent logistics, convergent IoT–AI–quantum frameworks and supply-chain resilience under digital transformation. Building on that foundation, theoretical models were constructed that map each of the above analytic ingredients onto concrete technological settings: spectral approximation of kernels, radial heat and load diffusion, spectral analysis of hierarchical load, continuum polar-flow models of infrastructure supply chains, and reduced-order modelling inside digital-twin architectures.

These results demonstrate that the classical theory of the Kontorovich–Lebedev transform continues to supply a powerful and rigorous language for multi-scale problems arising in modern computational and logistical infrastructures.

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