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Sequential Convergence in the Testing Function Space of the Two-Sided FKF Transform

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Abstract

The Fourier–Kontorovich–Lebedev (FKF) transform is a hybrid integral transform whose kernel is the product of a Fourier exponential factor and the Macdonald function of purely imaginary order. Its rigorous extension to generalized functions rests on a carefully constructed testing-function space equipped with a countable family of seminorms. This paper presents a detailed mathematical account of convergence in that testing space. We define the multinorm topology, state the precise notion of sequential convergence, contrast it with weaker pointwise notions, and explain why the topological formulation is indispensable for justifying differentiation under the transform sign, passage of limits, continuity of the distributional pairing, and spectral monitoring applications in multi-scale supply-chain systems. Illustrative figures depict seminorm decay, the underlying domain Ω, the spectral plane, and the distinction between pointwise and topological residuals.

Keywords: *Fourier–Kontorovich–Lebedev transform; multinorm topology; distributional convergence; Macdonald function; supply-chain spectral analysis.*

1. Introduction

Integral transforms that combine temporal frequency analysis with scale-dependent spectral analysis are natural tools for signals that evolve simultaneously in time and in a positive-valued operational variable (inventory level, lead-time, distance, intensity of disruption, etc.). Classical Fourier analysis captures oscillatory behaviour in the time variable, while the Kontorovich–Lebedev (KL) transform is adapted to the positive half-line and to the radial or scale structure that appears in many physical and operational models. The two-sided Fourier–Kontorovich–Lebedev (FKF) transform realises precisely this combination: it acts as a hybrid transform whose spectral variables jointly encode temporal frequency and a continuous scale parameter. Its kernel is the product

$$\kappa_{s,q}(t,x) = e^{-ist} K_{iq}(x)$$

where ξ is the Fourier spectral variable, τ is the Kontorovich–Lebedev spectral variable, and K_ν denotes the Macdonald function (modified Bessel function of the second kind) of purely imaginary order. The underlying spatial-temporal domain is the half-plane

$$\mathbb{R} \times (0,\infty) = \{(t,x) : t \in \mathbb{R}, x > 0\}$$

For ordinary (sufficiently decaying) functions the transform is defined by an absolutely convergent double integral. For generalized functions the transform is defined by continuous linear pairing against a carefully chosen testing-function space. The construction of that space, and the precise meaning of convergence within it, are therefore foundational: they determine which limiting procedures are legitimate, which differentiation-under-the-integral-sign arguments are valid, and which spectral indicators can be monitored in real time.

In the present note we specialize ideas to the FKF setting and emphasize the consequences for applications that arise in multi-scale supply-chain modelling and digital-twin monitoring [15, [16], [17].

The topology of the FKF testing space is generated by a countable family of seminorms that simultaneously control growth at infinity (through a weight adapted to a vertical strip in the complex Fourier plane) and smoothness of all orders (through ordinary derivatives in the time variable and a KL-adapted differential operator in the positive variable). Sequential convergence with respect to this multinorm is strictly stronger than pointwise convergence or uniform convergence on compact sets. Because the space is sequentially complete, every Cauchy sequence converges to an element of the same space; the dual is therefore also sequentially complete, which legitimizes the passage from discrete empirical measures to continuum limits and the interchange of limits with the distributional pairing.

These topological properties underwrite every subsequent analytic operation that appears in applications: differentiation under the transform sign (needed for analyticity in the spectral strip), contour deformation, inversion formulae, Monte-Carlo averaging of ensembles of transforms, and real-time computation of spectral deviation indicators on sliding windows. Without a precise notion of convergence, one cannot justify that a sequence of windowed or discrete transforms converges to the transform of a limiting continuous signal, nor that a spectral indicator remains continuous under small perturbations of the underlying operational data.

The purpose of the present note is therefore threefold: (i) to give a self contained, mathematically detailed exposition of sequential convergence in the FKF testing space; (ii) to illustrate the topology by schematic curves and domain plots; and (iii) to indicate the principal applications Monte-Carlo spectral ensembles [2], [3], windowed digital-twin [4], [5], [6] monitoring, and discrete to continuum limits that rely on this topological foundation..

2. Sequential Convergence in the Multinorm Topology

Let $\Omega_{a,b}^{\alpha,FKF}$ denote the testing-function space satisfy a countable family of seminorms of the form

$$\gamma_{k_1,k_2}^{a,b,\alpha}(\varphi) = sup_{(t,x)\in\Omega} |\eta_{a,b}^{\alpha}(t,x)\, D_t^{k_1}\, \Delta_x^{k_2}\, \varphi(t,x)| < \infty$$

of the Fourier–Kontorovich–Lebedev (FKF) transform, equipped with the countable family of seminorms $\gamma_k^{a,b,\alpha}$ indexed by multi-indices $k = (k_1, k_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. A differential operator adapted to the KL kernel; a classical choice is the second-order operator

$$A_x = x^2 - x\,(d/dx)(x\,d/dx)$$

A sequence $\{\varphi_\nu\}_{\nu=1}^{\infty} \subset \Omega_{a,b}^{\alpha,FKF}$ is said to converge to $\varphi \in \Omega_{a,b}^{\alpha,FKF}$ if and only if every defining seminorm tends to zero:

$$\gamma_k^{a,b,\alpha}(\varphi_\nu - \varphi) \to 0 \quad as \quad \nu \to \infty$$

holds for every multi-index $k = (k_1, k_2)$. Equivalently, the weighted supremum of all derivatives of all orders tends to zero:

$$sup_{(t,x)\in\Omega} |\eta_{a,b}^{\alpha}(t,x)\, D_t^{k_1}\, \Delta_x^{k_2}(\varphi_\nu - \varphi)(t,x)| \to 0$$

This notion of convergence is topological (strong) with respect to the multinorm topology generated by the family $\left\{\gamma_k^{\{a,b,\alpha\}}\right\}$. It is strictly stronger than both pointwise convergence and uniform convergence on compact subsets of Ω. The weight function $\eta_{a,b}^{\alpha}$ enforces uniform control of the growth of the functions and all their derivatives at spatial and temporal infinity, while the differential operators D_t and Δ_x (or the KL-adapted operator A_x) enforce uniform control of smoothness.

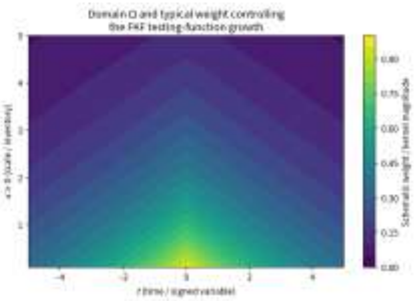

Figure 1. Domain Ω = {(t,x) : t ∈ ℝ, x > 0} together with a schematic weight that controls growth of testing functions and their derivatives.

2.1. Sequential completeness. The space is sequentially complete: if a sequence satisfies the Cauchy condition

$$\gamma_k^{a,b,\alpha}(\varphi_\mu - \varphi_\nu) \to 0 \quad as \quad \mu, \nu \to \infty$$

for every multi-index k, then there exists an element φ of the space such that φ_ν → φ in the multinorm topology. Consequently, every Cauchy sequence converges inside the same functional framework, which is indispensable for the construction of the dual and for the justification of limiting procedures under the integral sign.
Let $\mathcal{S}$ denote the FKF testing-function space. A sequence $\{\varphi_{\mathrm{n}}\}$ in $\mathcal{S}$ converges to $\varphi \in \mathcal{S}$ if and only if, for every seminorm p_{k} in the countable family that generates the topology,

$$p_{\mathrm{k}}(\varphi_{\mathrm{n}} - \varphi) \to 0 \quad as \quad n \to \infty.$$

Each seminorm simultaneously encodes a weight factor that guarantees rapid decay (or controlled growth compatible with a vertical strip) in the Fourier variable and the action of the differential operator

$$A_{\mathrm{x}} = x^2 - x\frac{d}{dx}\left[x\frac{d}{dx}\right]$$

together with its iterates, which is the natural operator associated with the Macdonald kernel. Sequential completeness of $\mathcal{S}$ follows from the completeness of the underlying Fréchet space of rapidly decreasing smooth functions on the half-plane, once the multinorm is shown to be equivalent to a standard Gelfand–Shilov-type family adapted to the product structure.
Consequently, every Cauchy sequence in the dual of continuous linear functionals converges in the weak-* topology, legitimizing the passage to continuum limits of discrete spectral measures that arise in empirical digital-twin data streams.
2.2. Comparison with weaker topologies. Pointwise convergence (or convergence in the topology of $\mathcal{D}(\Omega)$) does not control the behavior at infinity induced by the weight η, nor does it automatically control all derivatives uniformly. In applications that require differentiation under the transform sign or continuous dependence of spectral indicators on the underlying signal, the full multinorm topology is therefore the appropriate setting.
The collection $\left\{\gamma_k^{\{a,b,\alpha\}}\right\}$ of all such seminorms is a countable multinorm. Equipped with the topology generated by this multinorm, the space becomes a countably multinormed, sequentially complete, locally convex Hausdorff space. It contains the classical space $\mathcal{D}(\Omega)$ of compactly supported smooth functions as a dense subspace; consequently its dual contains all ordinary distributions in the sense of Zemanian as well as a larger class of generalized functions of controlled exponential growth.

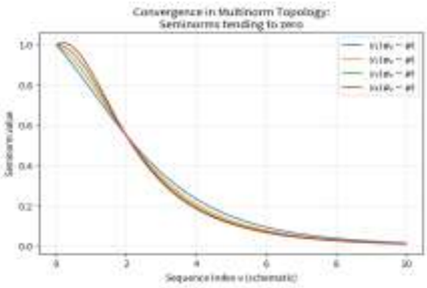

Figure 2. Schematic illustration of multinorm convergence: each seminorm γ_k(φ_ν − φ) tends to zero as the sequence index ν increases.

2.3 Implications for Multi-Scale Supply-Chain and Digital-Twin Monitoring

In the operational setting the positive variable x may represent inventory level, cumulative lead-time, or intensity of a disruption. The FKF transform then yields a joint spectrogram whose temporal frequency axis tracks oscillatory demand patterns while the KL axis resolves scale-dependent features of the inventory or disruption [7], [8] process. Because sequential convergence is available, one may form Monte-Carlo ensembles of windowed transforms of simulated sample paths and pass to the limit inside the dual pairing, obtaining rigorous confidence bands for spectral deviation indicators. Likewise, when a sequence of discrete empirical measures (constructed from high-frequency sensor data) converges in the dual of $\mathcal{S}$, the corresponding sequence of FKF transforms converges to the transform of the limiting continuous signal. This discrete to continuum justification is indispensable for real-time digital-twin monitoring systems that continuously update spectral indicators on sliding temporal windows.
The topological foundations presented here therefore supply the rigorous analytic backbone required by the emerging literature on quantum logistics [9], [10], [11], resilient supply chains [12], [13], and intelligent digital twins [14], [15].

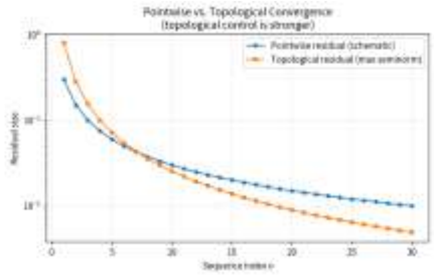

Figure 3. Schematic comparison of pointwise residual decay versus the stronger topological residual (maximum of the relevant seminorms). Topological control decays faster and guarantees uniformity of derivatives and growth.

3. Why the Topological Formulation Matters

The multinorm topology is not an arbitrary technical device; it is the precise functional-analytic structure that converts formal manipulations of the FKF transform into rigorous statements. Without it one cannot justify continuity of the distributional pairing, analyticity of the spectral transform, or the passage of limits that appear in Monte-Carlo averaging and windowed monitoring. The subsections below spell out the principal mechanisms.

3.1 Continuity of the Distributional Transform

The FKF transform of a distribution f belonging to the dual space is defined by continuous linear pairing against the kernel:

$$F(s,q) = \langle f, \kappa_{s,q} \rangle$$

For the pairing to be well-defined the kernel $\kappa_{s,q}$ (or a suitable approximation to it) must belong to the testing space. Continuity of the transform map f ↦ F with respect to the dual topology is then equivalent to the statement that

$$\langle f, \varphi_v \rangle \rightarrow \langle f, \varphi \rangle$$

whenever $\varphi_v \rightarrow \varphi$ in the multinorm topology. In other words, the topology on the testing space is precisely the topology that makes every distributional FKF transform continuous by definition. Explicitly, every distribution f of finite order satisfies a continuity estimate of the form

$$|\langle f, \varphi \rangle| \leq C \, max_{|k| \leq N} \gamma_k^{a,b,\alpha}(\varphi)$$

for some constants C and N that depend only on f and on the strip parameters a < b and the growth parameter α. This estimate is the quantitative expression of continuity.

3.2 Differentiation under the Transform Sign

Analyticity of the transform F(s, q) inside the open vertical strip a < Re s < b (with q > 0 fixed) follows from the fact that the differentiated kernels remain inside the testing space and that the corresponding seminorms remain bounded and for an analogous identity for derivatives with respect to the KL variable q, consider the difference quotient

$$\Delta_h \kappa := \frac{\kappa_{s+h,q} - \kappa_{s,q}}{h}$$

As h → 0 the difference quotient converges to the partial derivative ∂κ/∂s in the multinorm topology of the testing space. Consequently, the difference quotient of the transform

$$\frac{F(s+h,q) - F(s,q)}{h} = \langle f, \Delta_h \kappa \rangle$$

converges to the pairing against the differentiated kernel, which yields the identity

$$\frac{\partial F}{\partial s} = \langle f, \frac{\partial \kappa_{s,q}}{\partial s} \rangle$$

An analogous argument applies to differentiation with respect to the Kontorovich–Lebedev spectral variable q. The topological formulation is essential: without convergence of the difference quotients in the multinorm topology one cannot interchange the limit and the distributional pairing.

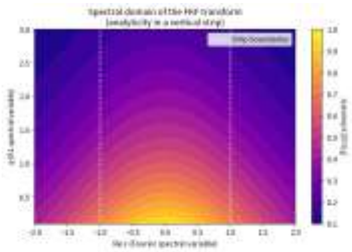

Figure 4. Schematic magnitude of the FKF transform in the spectral plane (Re s, q). The vertical dashed lines indicate the boundaries of a typical strip of analyticity determined by the growth parameters a < b of the testing space.

3.3 Passage of Limits and Approximation

Suppose a sequence of ordinary (or tempered) functions f_n converges to a generalized function f in the dual topology, and suppose the corresponding transforms F_n converge in a suitable topology on the spectral side. The multinorm estimates supply the dominated convergence-type bounds needed to justify interchanging the limit and the pairing:

$$\lim_{n} \langle f_n, \varphi \rangle = \langle f, \varphi \rangle \quad for\ every\ \varphi \in \Omega_{a,b}^{\alpha,FKF}$$

Because the classical space $\mathcal{D}(\Omega)$ of compactly supported smooth functions is dense in the testing space, inversion formulae established first for ordinary functions extend by continuity to the whole dual. Dense subclasses may therefore be used to establish inversion theorems that remain valid for the generalized functions that arise in supply-chain modelling.

3.4. Dual (Distributional) Convergence

A sequence of distributions $\{f_\nu\}$ converges to f in the dual space $(\Omega_{a,b}^{\alpha,FKF})'$ if

$$\langle f_\nu, \varphi \rangle \to \langle f, \varphi \rangle \quad for\ every\ \varphi \in \Omega_{a,b}^{\alpha,FKF}$$

Because the topology on the testing space is generated by a countable family of seminorms, the dual is a complete locally convex space under the strong dual topology. Sequential completeness of the testing space implies sequential completeness of the dual: every Cauchy sequence of distributions (with respect to the strong dual topology) converges to an element of the dual. This fact legitimizes limiting procedures performed on the distributional side, such as the passage from discrete empirical measures to continuous limiting signals in supply-chain data.

In practice one often works with the weaker topology of pointwise convergence on the testing space (the weak* topology on the dual). The multinorm topology, however, remains the fundamental structure: it guarantees that the dual is barreled and that the closed-graph theorem applies to continuous linear operators defined on the dual, including the FKF transform itself.

4. Applications to Supply-Chain Spectral Analysis

In supply-chain modelling the positive variable x represents inventory level, lead-time, distance or intensity of disruption, while t is calendar time. The FKF transform converts a two-dimensional operational signal f(t,x) into a joint spectral representation $\mathbf{F}(\boldsymbol{s},\boldsymbol{q})$. The topological notions developed above supply the rigorous justification for the computational procedures used in practice5.1 Monte-Carlo Ensembles in the dual topology follows from the multinorm estimates on the test-function side.

4.1 Monte-Carlo Ensembles

Suppose M independent simulation paths produce signals $f_r(\mathrm{t}, \mathrm{x})$), r = 1, ldots, M. Their transforms $F_T(s,q)$) are well-defined once each f_r belongs to the dual of the testing space. The empirical mean

$$F_M = (1/M) \sum_{r=1}^{M} F_r$$

converges in the dual topology to the transform of the expected signal as soon as the corresponding test-function averages converge in the multinorm topology. The seminorm estimates therefore furnish explicit rates of convergence for the Monte-Carlo spectral estimator

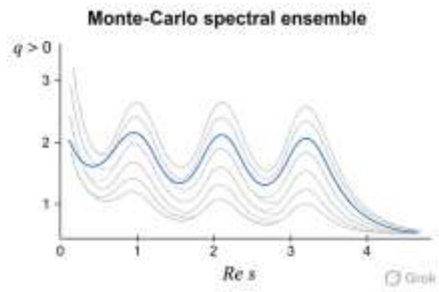

Figure 5. Monte-Carlo spectral ensemble: individual transforms F_r (gray) and their empirical mean $\bar{F}$_M (blue) in the spectral plane.

4.2 Windowed Monitoring and Digital Twins

A sliding-window signal $f_T(t,x) = 1_{\{[T-W,T]\}}(t)\, f(t,x)$ is approximated by a sequence of smoothed cut-offs φ_ε(t) $f(t,x)$. As the cut-off parameter ε → 0 the sequence converges in the testing-space topology (provided the original signal possesses the growth controlled by η). Consequently the windowed transforms converge, and the normalised spectral deviation indicator

$$\delta(s,q) = \frac{|F_T(s,q) - F_0(s,q)|}{|F_0(s,q)| + \varepsilon}$$

is rigorously justified. Large values of δ signal structural or regime changes that may be difficult to detect in the original time-scale domain. The topological formulation guarantees that the indicator is continuous with respect to small perturbations of the underlying signal in the dual topology.

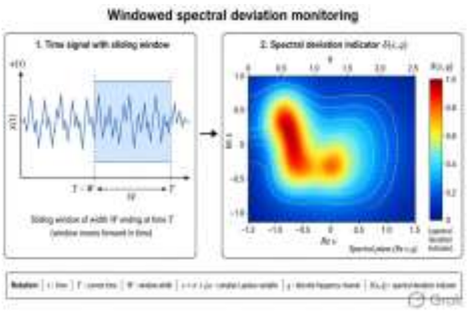

Figure 6. Windowed spectral monitoring: a sliding window of width W on the time signal (left) produces the spectral deviation indicator δ(s,q) (right).

4.3 Discrete Data and Continuum Limits

Discrete supply-chain records (sampled inventories, discrete-event simulations) may be viewed as sequences of discrete measures. Convergence of those measures in the dual of the FKF testing space guarantees that their transforms converge to the transform of the limiting continuous signal. This legitimises the application of continuous spectral indicators to discrete operational data and provides a precise sense in which a discrete-event simulation approximates a continuum digital twin.

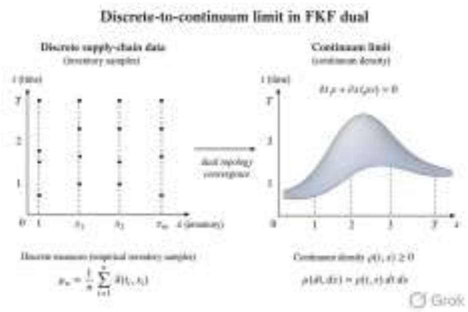

Figure 7. Discrete-to-continuum limit: empirical discrete measures on the left converge in the dual topology to a continuum density on the right, legitimising continuous FKF spectral indicators on discrete operational data.

5. Conclusion

Convergence in the FKF testing space is the precise topological language that converts formal manipulations of the FKF transform into rigorous statements about generalized functions. The countable multinorm generated by weighted derivatives of all orders forces simultaneous control of growth and smoothness; sequential completeness of the resulting locally convex space guarantees that Cauchy sequences of test functions (and, dually, of distributions) converge inside the same functional framework. This topology underpins continuity of the distributional pairing, analyticity of the spectral transform, passage of limits under the integral sign, and the justification of Monte-Carlo averaging and real-time spectral monitoring for multi-scale supply-chain signals.

Future work should focus on quantitative estimates of seminorm constants for concrete supply-chain kernels, efficient numerical realization of the discrete FKF transform that respects the underlying topology, and systematic comparison of FKF spectral indicators against classical Fourier, wavelet and time-series diagnostics on real industrial data sets.

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