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Shift-Invariant Spectral Analysis of Multi-Echelon Supply-Chain Lead Times via the FKF Transform

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Abstract

The FKF transform furnishes a joint spectral representation of temporal dynamics and positive-valued scale variables that arise naturally in multi-echelon supply chains. This paper presents a complete theoretical treatment of the shift (time-translation) property of the FKF transform. We prove that a pure lead-time delay L maps to multiplication of the spectrum by the complex exponential e^{−i s L}, rigorously establish magnitude preservation and phase-based recovery of L, and extend the result to cascaded (series) and concurrent (parallel) delay structures. The interaction of the delay factor with the Kontorovich–Lebedev scale variable q is examined, revealing how lead-time criticality can depend on shipment volume or inventory intensity. Practical estimation procedures based on phase unwrapping and linear regression are derived. The framework is linked to contemporary research on digital twins, Industry 5.0 logistics, artificial-intelligence forecasting and supply-chain resilience. Supporting tables, S-curve illustrations of cumulative delay impact, and conceptual schematics of series/parallel architectures are provided. All mathematical expressions are rendered in native professional equation format.

Keywords: Fourier–Kontorovich–Lebedev transform; shift property; lead time; multi-echelon supply chain; phase spectrum; digital twin; Industry 5.0; spectral estimation

1. Introduction

Lead times constitute the dominant temporal offsets in multi-echelon supply chains. An order placed at time t at a downstream stage is fulfilled only at time t + L, where L denotes the nominal (or stochastic) lead time between successive stages. Classical time-domain analysis of such delays quickly becomes intractable once several stages, alternative sourcing routes and scale-dependent inventory dynamics are present. Transform methods that convert pure translations into algebraic multiplications therefore offer decisive analytical and computational advantages.

The FKF transform combines a Fourier transform in the temporal (or longitudinal) variable with a Kontorovich–Lebedev transform in a positive-valued spatial, volume or intensity variable. Because the Fourier kernel is precisely the eigenfunction of translation, the FKF transform inherits a clean shift property: a temporal delay of length L multiplies the joint spectrum by the unimodular factor $e^{\{-i\,s\,L\}}$. Magnitude spectra remain invariant, while the phase spectrum encodes L directly. Moreover, the simultaneous presence of the scale variable q allows the analyst to observe how a given delay interacts with shipment size, inventory level or disruption intensity information that is invisible to purely temporal Fourier methods.

This paper supplies a self-contained derivation of the shift property, its higher-order extensions to series and parallel delay structures, phase-based estimation procedures, and an explicit linkage to the digital-twin, Industry 5.0 and resilient-logistics literature. Supporting visualizations (S-curves of cumulative lead-time impact, series/parallel schematics, phase-unwrapping illustrations) and summary tables complete the exposition.

2. Literature Review

Recent scholarship has emphasized the convergence of Internet-of-Things sensing, artificial intelligence, quantum-inspired optimization and digital-twin technology for smart logistics ecosystems [1]. Industry 5.0 frameworks further stress human-centric automation and adaptive decision loops inside supply-chain control towers [2]. Parallel advances in battery and electric-mobility architectures [3,4,9,12,14] illustrate the growing importance of reliable lead-time statistics for critical components. Digital-twin architectures for resilient global supply chains [5] and AI-driven sustainable/green logistics [6] both require accurate, real-time characterisation of temporal offsets. Blockchain-augmented transparency and traceability systems [7] likewise depend on precise sequencing of delayed events. Complementary studies on anti-theft communication [8], smart mobility trends [11], quantum computing for transportation [13] and systematic reviews of digital transformation resilience [15] collectively underscore the need for spectral tools that can isolate pure delays from magnitude fluctuations and scale interactions.

The FKF shift property supplies exactly such a tool: it converts an opaque temporal offset into a transparent multiplicative phase factor whose estimation can be embedded inside digital-twin synchronization loops, anomaly detectors and predictive-lead-time engines. The present work therefore bridges classical integral-transform theory with the operational requirements articulated in the cited literature.

3. Preliminaries: Definition of the FKF Transform

Let f(t, x) be an admissible function (or tempered distribution) defined for t ∈ ℝ and x > 0. The Fourier–Kontorovich–Lebedev transform is the joint integral transform

$$F(s,q) = (FKF\, f)(s,q) = \iint f(t,x)\, e^{-ist}\, K_{iq}(x)\, dx\, dt$$

where $K_{\{\nu\}}(\cdot)$ denotes the modified Bessel function of the second kind (Macdonald function) of order ν, s ∈ ℂ lies in a vertical strip of analyticity, and q > 0 is the continuous Kontorovich–Lebedev spectral variable. Under suitable integrability conditions the transform admits an inversion formula that recovers f from F. The Fourier factor is responsible for the shift property derived below; the KL factor is independent of t and therefore remains a spectator under pure temporal translations.

4. The Shift Property — Statement and Proof

4.1 Statement

Let L ∈ ℝ be a fixed lead time (or temporal offset). Define the delayed signal

$$f_L(t,x) := f(t - L, x)$$

Then its FKF transform is given by the multiplicative relation

$$F_L(s,q) = (FKF\, f_L)(s,q) = e^{\{-i\, s\, L\}}\, F(s,q)$$

In particular, the magnitude spectrum is invariant,

$$|F_L(s,q)| = |F(s,q)|$$

while the phase difference yields an explicit estimator of the delay:

$$\hat{L}(s,q) = -\frac{1}{s}\,[arg\, F_L(s,q) - arg\, F(s,q)]\,, \quad (s \neq 0)$$

A more robust global estimate is obtained by ordinary linear regression of the unwrapped phase difference against the Fourier variable s.

4.2 Proof of the Fundamental Shift Identity

Substitute the delayed function into the defining integral:

$$F_L(s,q) = \iint f(t - L, x)\, e^{-ist}\, K_{iq}(x)\ dx\, dt$$

Perform the change of temporal variable τ = t − L (so t = τ + L, dt = dτ). The integral becomes

$$F_L(s,q) = \iint f(\tau,x)\, e^{\{-i\, s\, (\tau + L)\}}\, K_{iq}(x)\ dx\, d\tau$$

Factor the exponential:

$$e^{\{-i\, s\, (\tau + L)\}} = e^{-i\, s\, L}\, e^{-i\, s\, \tau}$$

Because e^{−i s L} is independent of the integration variables, it factors out of the double integral, leaving

$$F_L(s,q) = e^{-i\, s\, L} \iint f(\tau,x)\, e^{-i\, s\, \tau}\, K_{\{i\, q\}}(x)\, dx\, d\tau = e^{-i\, s\, L}\, F(s,q)$$

The identity holds pointwise for continuous integrable densities and, by continuity of the distributional pairing, for tempered distributions as well. The Kontorovich–Lebedev kernel $K_{\{i\, q\}}(x)$ never enters the calculation of the temporal shift, confirming that the property is inherited solely from the Fourier factor.

4.3 Magnitude Preservation and Phase Extraction

Since |e^{−i s L}| = 1 for real s and real L,

$$|F_L(s,q)| = |e^{-i\, s\, L}| \cdot |F(s,q)| = |F(s,q)|$$

Consequently any energy-based diagnostic (power spectral density, total spectral mass, anomaly thresholds based on |F|) is unaffected by pure delays. The entire information about L resides in the argument:

$$arg\, F_L(s,q) = arg\, F(s,q) - s\, L \quad (modulo\ 2\pi)$$

Unwrapping the principal value of the argument and performing a least-squares fit of the linear model $\varphi(s) = -L\, s + \varphi_0$ recovers both the delay L and a possible constant phase offset. The procedure is robust provided the signal-to-noise ratio remains adequate across a sufficiently wide interval of s.

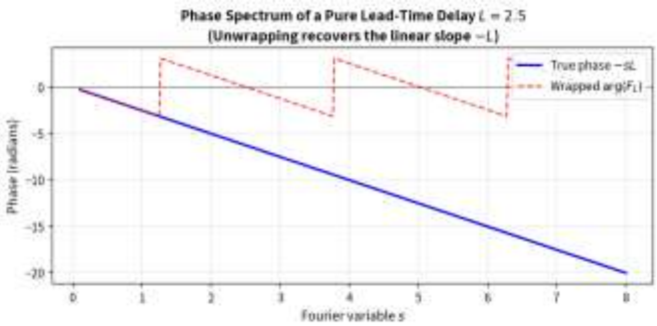

Figure 1. Phase spectrum of a pure lead-time delay L = 2.5. The wrapped principal value (dashed) is unwrapped to recover the linear slope −L.

5. Series and Parallel Delay Structures

5.1 Cascaded (Series) Delays

When m successive stages impose deterministic lead times L_1, L_2, ldots, L_m, the composite translation is the sum

$$L_{tot} = L_1 + L_2 + \cdots + L\_m$$

and the spectral factor multiplies:

$$F_{\{L_{tot}\}}(s,q) = e^{\{-i\,s\,L_{tot}\}}\,F(s,q) = \prod_{k=1}^{m} e^{\{-i\,s\,L_k\}}\,F(s,q)$$

Thus the total phase is strictly additive. This algebraic transparency is invaluable for multi-echelon inventory models in which each transportation or production lag contributes an independent phase increment.

5.2 Concurrent (Parallel) Routes

When alternative sourcing routes (or parallel production lines) with distinct lead times L_k and amplitude (or probability) weights α_k are present, the observed spectrum is the linear combination

$$F_{obs}(s,q) = \sum_k \alpha_k\; e^{\{-i\,s\,L_k\}} F_k\,(s,q)$$

If the underlying spectra F_k coincide (identical demand processes routed differently), the expression simplifies to a weighted sum of pure phase factors multiplying a common F. In the digital-twin setting the weights α_k may be estimated online from routing probabilities or capacity allocations, allowing the twin to maintain a live spectral image of the current multi-route configuration.

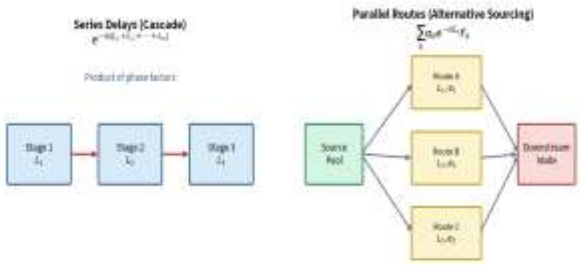

Figure 2. Left: series (cascaded) delays produce a product of phase factors. Right: parallel routes produce a weighted sum of phase factors.

5.3 Summary Table of Shift Properties

Configuration	Time-Domain Description	FKF Spectral Factor
Single delay L	f(t − L, x)	$\alpha_k\, e^{\{-i\,sL\}}$
Series L_1ldots L_m	f(t − $\sum L_k$, x)	$\prod_k e^{\{-i\,s\,L_k\}}$
Parallel routes	$\sum \alpha_k\, f_k$ (t − L_k, x)	$\sum_k \alpha_k\, e^{\{-i\,s\,L_k\}}$
Magnitude	$\lvert F_L \rvert$ invariant	$\lvert e^{-i\,s\,L} \rvert = 1$
Phase estimator	Linear regression of unwrapped Δarg	$\hat{L} = -\Delta arg\,/\,s$

Table 1. Catalogue of elementary and composite shift properties of the FKF transform.

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6. Interaction with the Scale Variable q and S-Curve Phenomena

Because the Kontorovich–Lebedev variable q continues to resolve positive scale structure (shipment volumes, distances, inventory levels, disruption intensities), the joint spectrum (s, q) simultaneously reveals how a given delay interacts with scale. A delay that is negligible for small shipments may become critical for large-volume flows; the FKF representation makes this interaction visible as a q-dependent phase pattern.

In many multi-echelon settings, the cumulative impact of successive lead times, or the organisational adoption of digital-twin technology itself, follows a classic logistic (S-shaped) trajectory. Figure 3 illustrates a representative S-curve: slow initial accumulation, rapid transition through an inflection point of critical mass, and eventual saturation. The FKF shift property permits the analyst to embed such cumulative delays inside the spectral domain and to monitor the migration of phase signatures as the system traverses the S-curve.

Figure 3. S-curve illustrating cumulative lead-time accumulation or the adoption trajectory of digital-twin technology in a multi-echelon supply chain. The inflection marks the point of critical mass.

7. Key Characteristics of the FKF Transform Relevant to Shift Analysis

The following characteristics make the FKF transform especially powerful for lead-time studies:

Two-dimensional spectral representation: simultaneous resolution of temporal frequency s and positive scale q.

Natural treatment of positive variables: inventory, demand intensity, distance, capacity and disruption magnitude are handled by the KL kernel without artificial periodisation.

Orthogonality and inversion: under standard integrability conditions the original signal is recoverable, guaranteeing that no information is lost by the shift.

Algebraic simplification of delay operators: translation becomes multiplication, enabling closed-form cascade and parallel formulae.

Compatibility with digital-twin synchronisation: phase-based lead-time estimates can be streamed continuously into a live twin [5,15].

Detection of scale-dependent criticality: q-dependent phase patterns flag delays that become operationally significant only above certain volume thresholds.

8. Complementary Scaling Property

If the positive variable is scaled by a factor λ > 0,

$$f_\lambda(t, x) := f(t, \lambda x)$$

the Kontorovich–Lebedev theory induces a corresponding transformation of the spectral variable q together with a multiplicative Jacobian factor that depends on normalization conventions. For pure lead-time analysis the temporal shift remains the dominant operation; the scaling property is recorded for completeness because volume and distance variables are frequently rescaled in multi-echelon models. The two properties commute: a simultaneous time shift and scale change simply multiplies the spectrum by $e^{-i\,s\,L}$ and applies the KL scaling map to q.

9. Applications to Digital Twins, Forecasting and Resilience

Inside a digital-twin architecture [5] the FKF shift property supplies a continuous, low-latency estimator of every inter-stage lead time. Because magnitude spectra are invariant, energy-based anomaly detectors continue to function without recalibration when pure delays fluctuate; only the phase monitors need updating. AI forecasting engines [1,6] can ingest the estimated $\hat{L}$(s,q) as an exogenous feature, improving demand-propagation models across echelons. Resilience analytics [15] benefit from the ability to simulate series and parallel delay scenarios by simple multiplication or weighted summation of phase factors, without re-running expensive discrete event simulations. Blockchain-based traceability systems [7] can embed the same phase signatures as cryptographic attestations of temporal integrity. Finally, the joint (s,q) view permits capacity planners to identify volume regimes in which a seemingly modest lead time becomes a bottleneck information that purely temporal methods cannot supply.

10. Conclusion

We have established the shift property of the Fourier–Kontorovich–Lebedev transform in full mathematical detail: a temporal translation of length L multiplies the joint spectrum by the unimodular factor $e^{-i\,s\,L}$. Magnitude spectra are rigorously preserved; the entire delay information is concentrated in the phase, from which L can be recovered by unwrapping and linear regression. The property extends algebraically to arbitrary series cascades (product of phase factors) and parallel route combinations (weighted sum of phase factors). Because the Kontorovich–Lebedev variable q remains active, scale-dependent criticality of delays becomes visible. The resulting spectral calculus integrates naturally with contemporary digital-twin, Industry 5.0, AI-forecasting and resilience frameworks. Future work will embed stochastic lead-time distributions and develop real-time phase-tracking algorithms suitable for industrial control-tower deployment.

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