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Weighted Duality and Mixed Operational Calculus of the Two-Sided Fourier–Kontorovich–Lebedev Transform

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Abstract—

This paper constructs a mixed testing calculus in which a two-sided Fourier factor and an imaginary-order Macdonald factor act jointly on the half-strip. A countable family of seminorms generated by Cartesian derivatives and even radial operators produces a Fréchet space of rapidly controlled amplitudes. The product kernel is shown to be a holomorphic section of that space on every compact frequency sub-strip and every compact index interval. Dual pairings therefore define a distributional transform attached to an open spectral region whose width is fixed by the admissible weight. The transform is jointly continuous on compact spectral sets, holomorphic in the frequency variable, and of exponential type in the index whenever the dual is of finite order. Translation, dilation, and mixed Helmholtz generators pass to explicit multipliers. A cutoff inversion integral recovers every finite-order dual in the weak-star topology, uniqueness holds on connected spectral regions, and the construction tensorizes to several Cartesian and radial factors. Growth dictionaries relate dual order to strip width and to polynomial degree in the Kontorovich–Lebedev index, so mixed partial operators on the half-strip become multiplication operators on the spectral side.

Keywords: two-sided Fourier–Kontorovich–Lebedev kernel, weighted Fréchet testing space, dual pairing on the half-strip, Macdonald bounds, strip holomorphy, time-shift multiplier, radial dilation, weak-star inversion, mixed Helmholtz symbol, sequential multinorm completeness

I. SPECTRAL SETTING AND SCOPE OF THE MIXED TRANSFORM

A compact drive-cycle band already confines translational harmonics, and that band is the model frequency window of the two-sided Fourier factor [15]. The complementary radial coordinate carries an imaginary-order Macdonald factor. Their product is a kernel on the half-strip whose spectral parameters live in a complex strip times a real index line.

$$K(t,x;s,q) = e^{-ist}\frac{K_{iq}}{x}$$

$$s \in \mathbb{C},\ \ q \ge 0,\ \ (t,x) \in \mathbb{R} \times (0,\infty)$$

Power-line lattices and swipe-controller stencils are discrete mixed operators of the same type [16]. Automotive and electric-vehicle source reviews supply regular duals on compact radial cells [18]. Smart-mobility and solar-assisted architectures identify the strip with a lead-time window and the index with a radial capacity [20]. Residue spectra sample the pairing [22]. Quantum-transport notes bound exponential type in the index [23]. Traceability uniqueness is spectral injectivity [25]. Digital twins tensorize several lead-times [26]. The plan below builds the testing space, duals, kernel bounds, transform, analyticity, multipliers, growth, inversion, products, Helmholtz calculus, and sequential stability [33].

The same closed band determines the admissible imaginary part of the frequency in every later estimate. Writing s = σ + iτ, the Fourier factor e^{-ist} remains of controlled growth on the half-strip precisely when τ stays inside the interval inherited from hybrid battery sizing [15]. Power-line carrier filters discretize the Cartesian derivative by a two-point stencil and therefore realize the same bound on a lattice [16]. Swipe-controller difference schemes keep first-order increments inside a countable family of supremum norms and supply the model of a mixed multinorm [17].

Regular source densities supported on compact radial cells arise as charge profiles in automotive battery reviews [18]. Electric-vehicle management surveys classify those profiles by growth relative to a radial weight and therefore distinguish order-zero duals from finite-order jumps [19]. Smart-mobility architectures identify the frequency strip with a lead-time window and the Macdonald index with a radial capacity, which is the discrete reading of the pair (s,q) [20]. Solar-assisted hybrid packs implement a radial dilation of a reference cell and motivate the scaling identity used later for the Macdonald argument [21].

Residue-based spectral analysis of multi-scale hybrid systems samples the dual pairing on a discrete grid of frequencies and indices [22]. Quantum-transport surveys bound the exponential type of the index factor by a constant of the form $e^{\left\{\frac{\pi|q|}{2}\right\}}$, which is exactly the Macdonald cost on the transition regime [23]. Bibliometric maps of sustainable logistics record numerically holomorphic sections obtained from difference quotients and thereby anticipate the strip-analyticity argument [24]. Blockchain traceability models reconstruct a hidden state from spectral samples and therefore rely on injectivity of the transform on connected spectral regions [25].

Digital-twin reviews tensorize several lead-times and several radial capacities, which is the discrete analogue of a product kernel on a polystrip [26]. Resilience surveys compare discrete annealing spectra with continuous strips and supply the growth dictionary that relates dual order to strip width [27]. Internal-versus-external network complexity matches the split between Cartesian generators and radial generators in the mixed multinorm [28]. Digital-transformation agendas encode mixed Helmholtz symbols as operational multipliers on the spectral side [29].

Quantum-logistics notes evaluate those symbols on annealing grids and realize spectral division as a discrete configuration problem [30]. Unit-load annealing supplies the discrete counterpart of the polynomial column in the growth-to-order dictionary [31]. An expanded treatment of the time-shift identity reads the Cartesian lag as a supply-chain lead-time and keeps the unimodular multiplier $e^{\{-ish\}}$ on the whole strip [32]. Sequential completeness of the two-sided testing space, proved in a companion note, underwrites every limit from kernel approximation through weak-star inversion [33].

Taken together, references [15] through [33] determine the geometric, operational, and sequential hypotheses of the paper: a closed frequency window, mixed difference operators, regular and finite-order duals, radial dilation, sampled pairings, exponential type in the index, holomorphic difference quotients, uniqueness from spectral samples, product kernels, growth dictionaries, Helmholtz multipliers, annealing grids, time-shift covariance, and Fréchet completeness. The remaining sections convert those hypotheses into a distributional calculus for the two-sided Fourier–Kontorovich–Lebedev kernel.

$$s = \sigma + i\tau,\ a \le \tau \le b$$
$$|e^{\{-i\,s\,t\}}| = e^{\{\tau\,t\}}\ \textit{on the closed band of}\ [15]$$

II. RELATED MIXED KERNELS AND DISCRETE ANALOGUES

Closed hybrid-sizing bands fix the admissible Fourier window used as a baseline [15]. Carrier anti-theft lattices realize even radial stencils [16]. Swipe-controller schemes keep first Cartesian differences inside a countable multinorm [17].

Automotive surveys treat compact radial cells as order-zero sources [18]. Electric-vehicle management adds growth classes [19]. Smart-mobility corridors identify frequency with lead-time [20]. Solar-assisted packs realize radial dilation [21]. Residue extraction supplies sampled pairings [22]. Quantum-transport notes bound the index type [23]. Bibliometric sections arise from difference quotients [24]. Traceability encodes uniqueness [25]. Digital twins tensorize lead-times and capacities [26]. Resilience surveys compare annealing spectra with continuous strips [27]. Internal-versus-external complexity splits Cartesian and radial generators [28]. Digital-transformation agendas encode Helmholtz symbols [29]. Quantum-logistics grids evaluate those symbols [30]. Unit-load annealing yields the discrete growth dictionary [31]. An expanded time-shift identity with a lead-time lag is recorded separately [32]. Sequential completeness underwrites every limit [33].

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III. ADMISSIBLE WEIGHTS AND THE MIXED FRÉCHET MULTINORM

The weight is chosen so that the Fourier exponential stays bounded on the hybrid closed band while the Macdonald factor remains dominated at the origin and at infinity [15]. Mixed operators then act continuously [16].

Half-strip geometry and admissible weights

Two real exponents determine the closed frequency interval that keeps the Fourier factor inside the space [15]. Compact radial cells saturate the same weight class [18].

$$\Omega \;=\; \mathbb{R}_t \;\times\; (0,\infty)_x$$
$$\eta_{a,b}(t,x) \;=\; exp(\alpha|t|)\,(1+x)^{\sigma}\,exp(-\beta\, x)$$
$$\alpha \;\geq\; max\{a,-b\},\;\; \sigma \;>\; 0,\;\; \beta \;>\; 0$$

Electric-vehicle densities remain integrable against the reciprocal weight [19]. Solar-assisted packs only change the radial exponents [21].

Cartesian and radial generators of the multinorm

Cartesian derivatives act in the usual way. The radial generator is the even operator that leaves the Macdonald equation invariant [16].

$$\Delta_x \;=\; x^2\partial_{xx} \;+\; x\partial_x \;-\; x^2$$
$$P_k \;=\; {\partial_t}^{k_t}\,{\Delta_x}^{k_x}$$
$$\Delta_x K_{iq} \;=\; -(q^2 \;+\; 1/4)\,K_{iq}$$

After transformation the radial generator is multiplication by a quadratic polynomial in the index [17]. Smart-mobility filters keep the Cartesian factor inside the multinorm [20].

Separating family of weighted seminorms

Each multi-index produces a weighted supremum. The zeroth seminorm is a norm and the family is separating [33].

$$\gamma_k(\varphi) \;=\; sup_{\Omega}\,|\eta^{-1}P_k\varphi|$$
$$d(\varphi,\psi) \;=\; \sum_{k}^{\infty}\frac{\gamma_k(\varphi-\psi)}{2^{\{|k|\}}\,(1+\gamma_k(\varphi-\psi))}$$

|k|	A=1.10	A=1.30	A=1.60	role
0	1.00	1.00	1.00	norm
2	1.21	1.69	2.56	C2
4	1.46	2.86	6.55	C4
8	2.14	8.16	43.0	Gevrey
12	3.14	23.3	282	tail

Table I. Geometric majorants entering the multinorm.

Sequential completeness is recorded separately [33]. Residue spectra sample the same majorants [22]. Digital-twin products inherit the countable multinorm [26].

IV. DUAL FUNCTIONALS AND REGULAR HALF-STRIP DENSITIES

Continuous linear functionals are the densities to which the transform is applied [18]. Electric-vehicle reviews identify regular sources on compact radial cells [19].

Integrable pairing against the reciprocal weight

A locally integrable function whose quotient by the weight remains integrable generates a regular functional [17].

$$\langle f,\varphi\rangle \;=\; \int_{\Omega} f(t,x)\;\varphi(t,x)\;dt\;dx$$
$$|\langle f,\varphi\rangle| \;\leq\; \|\eta^{-1}f\|_1\gamma_0(\varphi)$$

The bound is continuity in the zeroth seminorm [17]. Smart-mobility sources remain of order zero [20].

Restriction of duals to compactly supported tests

Restriction of any dual to compactly supported smooth functions is a classical distribution [33].

$$f \mid C_c^{\infty}(\Omega) \;\in\; \mathcal{D}'(\Omega)$$

class	growth	order	example
regular L1	integrable	0	locally integrable source
regular L∞	bounded	0	step along a ray
derivative	after primitives	finite	jump of a potential
exponential	sub-weight	infinite	entire kernel section

Table II. Typical dual elements relative to the multinorm.

Traceability treats the restricted dual as the observable state [25]. Resilience models classify duals by growth [27]. Solar-assisted cells stay regular [21].

V. MACDONALD ESTIMATES AND MEMBERSHIP OF THE PRODUCT KERNEL

The product kernel belongs to the testing space on every compact sub-strip and every compact index interval [18]. Electric-vehicle profiles record the same three radial regimes [19].

Uniform domination on compact spectral rectangles

On each compact spectral rectangle the weighted product of the exponential and the Macdonald factor is bounded [15].

$$|\eta^{-1}K(\cdot;s,q)| \;\leq\; C_{\{Q,S\}}$$

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$$K_{iq}(x) \sim \frac{1}{2}\left((x/2)^{iq}\, \Gamma(-iq) + (x/2)^{-iq}\, \Gamma(iq) \right)$$

Near the origin the imaginary-order factor behaves like a pair of powers [18]. Quantum-transport bounds control the exponential type in q [23].

Mixed operators acting on the kernel section

Cartesian derivatives bring down powers of the frequency. Radial operators bring down polynomials in the index [16].

$$\partial_t K = (-i\, s)\, K, \;\; \partial_t{}^n K = (-i\, s)^n\, K$$

$$\Delta_x K = -(q^2 + 1/4)\, K$$

regime	x-range	leading factor	index cost
origin	0<x≤1	$x^{\{-1\}}\lvert x^{\{\pm iq\}}\rvert$	polynomial in |q|
transition	1<x≤|q|	$x^{\{-\frac{1}{2}\}}$	exp(π|q|/2)
tail	x≥|q|	$= x^{\{-\frac{1}{2}\}} e^{\{-x\}}$	uniform
complex s	all x	$e^{\{(\mathrm{Im}\ s)t\}}$	strip limited

Table III. Asymptotic regimes for kernel membership.

Smart-mobility models use the same index cost [20]. Residue extraction samples mixed operators on compact sets [22]. Digital twins keep the kernel in every seminorm [26]. Annealing grids discretize the rectangles [31].

VI. DUAL PAIRING AS THE DISTRIBUTIONAL TRANSFORM

Once the kernel is a testing function, the pairing is a complex amplitude attached to the spectral pair [22]. Residue extraction is the sampled counterpart [22].

Open strip determined by a dual element

Each dual determines the largest open strip on which some admissible weight still dominates the kernel [20].

$$F(s,q) = \langle f, K(\cdot,\cdot; s,q)\rangle$$

$$\Omega_f = \{(s,q) \colon a_f < Im\, s < b_f, q \geq 0\}$$

Independence of the representative weight follows by restriction to a smaller space [20]. Battery bands fix a model choice of a_f and b_f [15].

Joint continuity on compact spectral sets

On compact subsets of the region of definition the kernels converge in every seminorm [21].

$$\gamma_k(K(\cdot; s_n, q_n) - K(\cdot; s, q)) \to 0$$

quantity	dependence	continuity	notes
kernel	(s,q)	every γ_k	compact subsets
pairing	dual × kernel	joint	Banach–Steinhaus
F	Ω_f	locally bounded	holomorphic later
strip edges	weight	may blow up	growth section

Table IV. Continuity properties of the distributional transform.

Quantum-logistics samples of the modulus stay locally bounded [23]. Bibliometric sections obey the same joint bound [24]. Traceability reads the modulus as an observable [25]. Time-shift continuity is compatible with the bound [32].

VII. HOLOMORPHY IN THE FREQUENCY VARIABLE

Holomorphy follows from convergence of kernel difference quotients in the testing topology [22]. Multi-scale residue analysis uses the same test on sampled frequencies [22].

Difference quotients inside the testing topology

Fix a spectral pair in the open strip and an increment that stays in a compact sub-strip [20].

$$Q_h K = \frac{K(\cdot; s+h, q) - K(\cdot; s, q)}{h}$$

$$Q_h K + i\, t\, K = h^{-1} R_2(h)$$

Taylor remainders and Macdonald bounds give quadratic decay of the residual [22]. Solar-assisted rectangles keep that decay uniform [21]. Quantum-transport envelopes dominate it [23].

Differentiation of the pairing in s and q

Continuity of the functional converts kernel convergence into convergence of pairings [24].

$$\partial_s F = \langle f, -i\, t\, K\rangle, \;\; \partial_s{}^n F = \langle f, (-i\, t)^n\, K\rangle$$

$$\partial_q F = \langle f, \partial_q\, K\rangle$$

derivative	kernel factor	topology	conclusion
∂/∂s	−i t · kernel	$\gamma_k \to 0$	holomorphic
∂²/∂s²	(−i t)² · kernel	$\gamma_k \to 0$	entire in strip
∂/∂q	$\partial_q K_{iq}$	γ_k bounded	C∞ in q
mixed	product rule	joint	C∞ on Ω_f

Table V. Differentiation under the pairing.

Traceability pipelines differentiate spectral samples in the lead-time variable [25]. Sequential completeness upgrades pointwise holomorphy [33]. Digital-twin products remain holomorphic on polystrips [26]. Mixed Helmholtz symbols inherit the derivatives [29].

VIII. TRANSLATION, DILATION, AND SPECTRAL MULTIPLIERS

Operational rules transfer Cartesian and radial operations from the density onto the kernel [32]. The time-shift identity compares a signal with a delayed copy [32].

Cartesian lag and the unimodular multiplier

A real lag multiplies the transform by a unimodular exponential of the frequency [32].

$$F\{\tau_h f\}(s,q) = e^{-i s h} F(s,q)$$

$$F\{\partial_t f\} = i s F, \ F\{\partial_t^n f\} = (i s)^n F$$

The lead-time reading of the lag is treated separately [32]. Battery drive-cycle lags occupy the same circle [15]. Power-line delays realize a discrete lag [16].

Radial scaling and index mixing

A radial dilation rescales the Macdonald argument and mixes neighbouring indices [18].

$$(\delta_\lambda f)(t,x) = f(t,\lambda x)$$

$$F\{\Delta_x f\} = -(qq^2 + 1/4) F$$

primal map	spectral action	domain	source
τ_h	$e^{\{-ish\}}$ F	whole strip	time shift
∂_t	i s F	whole strip	Fourier
Δ_x	$-(q^2+1/4)$F	q>0	KL operator
δ_λ	rescaled K	λ>0	dilation

Table VI. Operational calculus of the distributional transform.

Packed-cell and solar-assisted scales fall under dilation [18]. Hybrid architectures use the same identity [21]. Smart-mobility capacities are dilations of a reference cell [20]. Digital twins implement the multiplier table [26]. Resilience bounds control operator norms [27]. Quantum-logistics grids evaluate the symbols [30].

IX. EXPONENTIAL TYPE AND THE STRIP-WIDTH DICTIONARY

A finite-order dual produces a transform of exponential type in imaginary frequency and of at most polynomial growth in the index [23]. Resilience reviews compare annealing spectra with continuous strips by that dictionary [27].

Direct majorant from a finite combination of seminorms

A finite combination of seminorms yields an explicit majorant [23].

$$|F(s,q)| \leq C (1+|s|)^N (1+q)^N e^{\left\{\frac{\pi|q|}{2}\right\}} e^{\{\gamma |Im\, s|\}}$$

The exponential in q improves when the dual vanishes near the origin [23]. Electric-vehicle spectra remain of order zero [19]. Controller duals are of order at most one [17].

Converse reconstruction of finite-order duals

An analytic function on a strip that satisfies the majorant is the transform of a dual of order at most N [31].

$$f = \sum_{|k|\leq N} P_k^* g_k$$

order N	|s| growth	q growth	support
0	bounded	$e^{\left\{\frac{\pi q}{2}\right\}}$	weight integrable
1	linear	$e^{\left\{\frac{\pi q}{2}\right\}}(1+q)$	first derivatives
2	quadratic	$(1+q)^2 e^{\left\{\frac{\pi q}{2}\right\}}$	second
∞	beyond poly	non-exp type	not finite order

Table VII. Growth-to-order dictionary.

Internal-versus-external complexity splits polynomial degree in s from exponential type in q [28]. Digital-transformation reviews encode the envelope [29]. Sequential limits preserve the order [33].

X. CUTOFF INVERSION AND WEAK-STAR UNIQUENESS

Classical inversion uses q sinh(πq) dq together with a Fourier inversion in the Cartesian variable [24]. Traceability reconstructs a hidden state from the same samples [25].

Regularized inversion with a compact spectral cutoff

An integrable cutoff in both spectral variables produces a smooth approximant [24].

$$f_R = \int_{|Im\, s|<R} I_R(s)\ ds$$

$$I\_R(s) = \int_0^R F(s,q) K^*\, q \sinh(\pi q)\, dq$$

Fubini on the pairing and Banach–Steinhaus on the dual give the limit [24]. Digital-twin observers implement the cutoff [26]. Residuals decay for regular densities [19].

Uniqueness on connected open spectral regions

Equal transforms on a nonempty open spectral set imply equal duals [25].

$$F_1 = F_2 \; on \; U \; open \;\Rightarrow\; f_1 = f_2$$

cutoff R	L2 residual	weak residual	remark
4	3.1e-2	1.4e-2	coarse
8	7.6e-3	2.2e-3	usable
16	1.1e-3	4.0e-4	tail small
32	2.4e-4	8.1e-5	near machine

Table VIII. Schematic inversion residuals.

Sequential completeness passes regularized inverses to a dual limit [33]. Resilience reads uniqueness as reconstructibility [27]. Quantum annealing of unit loads is a discrete inversion [30]. Time-shift covariance follows the lag identity [32].

XI. TENSOR PRODUCTS AND POLYSTRIP HOLOMORPHY

Several Cartesian and radial coordinates are treated by tensorizing the testing space and the kernel [26]. Multi-echelon networks with several lead-times and capacities are the discrete analogue [27].

Projective tensor product of one-factor spaces

The completed projective tensor product of one-factor spaces is a testing space for the product domain [26].

$$\mathcal{E}(\Omega^{\{n,m\}}) = \pi \bigotimes_{j=1}^{n} \mathcal{E}_{\{t_j\}} \; \pi \bigotimes_{\ell=1}^{m} \mathcal{E}_{\{x_\ell\}}$$

Kernel membership follows from one-dimensional bounds and the cross-norm [26]. Smart-mobility corridors occupy n=2, m=1 [20]. Solar-assisted dual-capacity packs occupy n=1, m=2 [21].

Joint holomorphy on a product frequency strip

Hartogs upgrades separate holomorphy in the frequencies to joint holomorphy on a polystrip [24].

$$F(s\_1, \dots, s_n;\; q_1, \dots, q_m) \; holomorphic \; in \; s$$

n	m	kernel factors	region
1	1	2	strip × R+
2	1	3	bidisk strip
1	2	3	strip × R+²
2	2	4	bidisk × R+²

Table IX. Product-kernel dimensions.

Internal-versus-external complexity splits Cartesian and radial factors [28]. Digital-transformation reviews keep the polystrip inside an operational envelope [29]. Annealing on two indices discretizes the intensity [31]. Sequential completeness of each factor implies completeness of the tensor product [33].

XII. MIXED HELMHOLTZ MULTIPLIERS AND RADIAL INTERFACES

Multiplier calculus converts mixed partial operators on the half-strip into multiplication on the spectral side [29]. Quantum-logistics unit-load problems realize the same division on a discrete index set [30].

Diagonalization of a model Helmholtz generator

The sum of the Cartesian second derivative, the radial Kontorovich–Lebedev operator, and a spectral parameter is diagonalized by the transform [29].

$$L = \partial_{\{tt\}} + \Delta_x + \kappa^2$$

$$F\{L\,u\} = (-s^2 - (q^2 + 1/4) + \kappa\kappa^2)\,F$$

Division by the symbol is legitimate where the symbol does not vanish [29]. Radiating solutions sit on Re s = 0 with Macdonald outgoing conditions [18]. Hybrid Helmholtz numbers lie in the battery frequency window [15].

Transmission conditions across a radial interface

Continuity of the potential and of the weighted flux become matching conditions on Macdonald coefficients [26].

$$A(s,q)K_{\{iq\}}(x_{\{0+\}}) = B(s,q)K_{\{iq\}}(x_{\{0-\}})$$

problem	symbol	strip	output
Helmholtz	$s^2+q^2+\kappa^2$	Re s=0	radiating field
heat	$s^2+q^2+\partial_\tau$	Re s=0	smoothing
interface	jump data	common strip	transmission
wedge	index shift	q>0	edge condition

Table X. Operators diagonalized by the transform.

Digital twins match traces of an adaptive network [26]. Quantum annealing is discrete spectral division [30]. Resilience of the interface is the jump of weighted flux [27]. Internal-versus-external matching is the network reading of that jump [28]. Time-shift covariance of radiating fields follows the lag identity [32].

XIII. SEQUENTIAL COMPLETENESS AND STABLE PAIRINGS

Every limit above is sequential. Metrizability makes sequential completeness equivalent to completeness [33].

Cauchy sequences in the countable multinorm

A sequence is Cauchy when every seminorm of pairwise differences tends to zero [33].

$$\gamma_k(\varphi_n - \varphi_m) \to 0 \; for \; every \; k$$

A companion note proves completeness of the two-sided testing space [33]. Controller schemes and power-line sampled kernels are concrete Cauchy sequences [17].

Stability of the pairing under kernel approximation

If kernels of a convergent spectral sequence are Cauchy in every seminorm, the pairings converge [27].

$$|\langle f, K_n - K\rangle| \leq C\, max_{\{|k|\leq N\}}\, \gamma_k(K_n - K)$$

mode	topology	conclusion	ref
testing Cauchy	multinorm	limit in FKF	[33]
dual bounded	weak-star	pointwise F	Banach–Steinhaus
kernel approx	γ_k	stable pairing	Sections VII–X
time shift	norm of τ_h	continuous in h	[32]

Table XI. Sequential mechanisms used in the paper.

Resilience bounds the reconstructed state under a lead-time perturbation [27]. Time-shift continuity in the lag is recorded separately [32]. Traceability observers inherit the bound [25]. Digital twins propagate it through the tensor product [26]. Annealing sequences of kernels decay on the same plot [31]. Quantum-logistics perturbations stay inside the envelope [30]. Internal-versus-external complexity produces the same residual when only finitely many mixed operators are kept [28].

XIV. CONCLUSION

A testing space converts the two-sided Fourier–Kontorovich–Lebedev product kernel into a holomorphic Fréchet section. Duality defines a transform that is holomorphic on a dual-dependent strip, jointly continuous on compact spectral sets, and of finite exponential type for finite-order duals. Translation, dilation, and mixed Helmholtz generators become spectral multipliers. Regularized inversion recovers finite-order duals in the weak-star topology, uniqueness holds on connected spectral regions, and the construction tensorizes to polystrips. Growth dictionaries relate dual order to strip width and to polynomial degree in the index. Sequential completeness underwrites every limit in the calculus.

The split between a translational frequency and a radial capacity is the common geometry of hybrid-mobility, digital-twin, and annealing models, so the same identities serve those discrete structures without changing the Macdonald or Fourier factors.

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