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Exact Solutions of the KdV Equation — Comment on Ghosh's paper

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Abstract: By applying the travelling wave method within a symbolic computation framework (MathHandbook), new exact analytical solutions of the nonlinear KdV equation were derived. Both the defocusing and focusing regimes were analyzed, revealing a duality in their solution structures. Rigorous validation through symbolic substitution confirmed the correctness of the solutions, providing exact benchmarks for nonlinear dynamics. A relationship between the defocusing and focusing solutions was found, which reveals the observed symmetry directly from the mathematical structure of the PDE. The analytical structure and stability of the obtained solution branches were discussed, and the Appendix corrects errors in previously published results.

Keywords: symbolic computation; nonlinear evolution equations; KdV equation; exact solutions; focusing–defocusing duality; solitons.

1. Introduction

Nonlinear wave equations arise in diverse areas of physics, including solid state physics, plasma physics, fluid dynamics, and quantum field theory [1–4]. Depending on the sign of the nonlinear coefficient, these equations exhibit focusing or defocusing behaviour, leading to rich dynamical structures. A variety of analytical techniques—such as the inverse scattering transform, Bäcklund transformations, Malfliet's method, and Jacobi elliptic function expansions—have been employed to obtain soliton, multisoliton, and periodic wave solutions [5–13].

The Korteweg–de Vries (KdV) equation is a famous nonlinear partial differential equation (PDE) modelling weakly nonlinear long waves in shallow water. It is renowned for describing solitary waves (solitons) that maintain their shape. Its basic form balances nonlinearity with dispersion:

$$\left[y_t + c\, y\, y_x + y_{\{xxx\}} = 0 \ \{1\} \right]$$

The KdV equation is a foundational example of an integrable system, meaning it admits exact solutions, possesses infinitely many conservation laws, and can be solved via the inverse scattering method. Applications range from plasma physics to cosmology. Zhang [14] used the Exp-function method to obtain exact solutions of the Klein–Gordon equation. He, Li, and Long [15] applied a modified Exp-function method to derive generalized wave solutions. Joseph [16] studied a general Klein–Gordon equation with quadratic nonlinearity and derived new exact solutions expressed in terms of Jacobi elliptic functions, though their validation is cumbersome. Ghosh [17] reviewed nonlinear systems and derived soliton solutions of the KdV equation; however, his solutions of Eqs. (11) and (12) are incorrect.

To addtress this issue, we explain why those solutions are invalid and employ symbolic computation to obtain new exact analytical solutions of a nonlinear wave equation, verifying the solutions by direct substitution. The analysis reveals structural correspondences between the focusing and defocusing regimes, demonstrating that the observed symmetry follows directly from the mathematical structure of the governing PDE.



2. Method

Several forms of the KdV equation have been studied in the literature [1–17]. The version with quadratic nonlinearity is given by Eq. (1).

For simplicity, we set (c = ± 2), distinguishing the defocusing (c = 2) and focusing (c = −2) cases:

$$[\, y_t \pm 2\, y\, y_x + \, y_{\{xxx\}} = \, 0 \;\{2\}]$$

Seeking a travelling wave solution, we introduce:

$$[\, Y(u) = \, y(t,x), \;\; u \, = \, x \, - \, t \;\{3\}]$$

Substitution into Eq. (2) reduces the PDE to an ordinary differential equation (ODE). For c = 2, we obtain:

$$[\, Y''' + \, 2\, Y\, Y' - \, Y' = \, 0 \;\{4\}]$$

with first integral:

$$[\, Y'' + \, Y^2 - \, Y \, = \, C \;\{5\}]$$

3. Results and Discussion

Symbolic computation provides exact benchmarks for nonlinear dynamics [6]–[8], qualitative insight into the solution structure [3, 4], and educational accessibility [1, 2]. It enables automated derivation of closed-form solutions of nonlinear PDEs, provides rigorous checks via symbolic algebra, supports multi-dimensional analysis, and reveals structural relationships such as the symmetry-induced duality between defocusing and focusing solutions.

3.1 Exact Solutions

The general solution of the second-order ODE can be expressed in terms of the Weierstrass elliptic function:

$$[\, Y(u) = \, 1/2 - \, 6\wp(C_1 \pm u;\; g_2, g_3) \;\{6\}]$$

Thus, the PDE solutions for c = 2 are:

$$[\, y_+(t,x) = \, 1/2 - \, 6\wp(C_1 \pm (x \, - \, t);\; g_2, g_3) \;\{7\}]$$

Similarly, for c = −2:

$$\left[\, y_-(t,x) = \, -\frac{1}{2} + \, 6\wp(C_1 \pm (x \, - \, t);\; g_2, g_3) \;\{8\}\right],$$

If $g_3 = \, 0$ and $g_2 = \, 0,$ Eqs. (7) and (8) reduce to rational soliton waves:

$$\left[\, y_+(t,x) = \frac{1}{2} - \frac{6}{\left\{\left(C_1 \pm (x \, - \, t)\right)^2\right\}} \;\{9\}\right]$$

$$[\, y_-(t,x) = \, -\frac{1}{2} + \frac{6}{\left\{\left(C_1 \pm (x \, - \, t)\right)^2\right\}} \;\{10\}\,]$$

Comparison of these solutions reveals a duality:

$$[\, y_+(t,x) = \, -\, y_-(t,x) \;\; \{11\}]$$

3.2 Validation

The equations were solved and the solutions were verified by direct substitution into Eq. (2) using the online symbolic computation system MathHandbook [20-23]. The framework confirmed exact cancellation, thereby validating the analytical results. In contrast, substituting Ghosh's solutions [17] of Eqs. (11) and (12) into Eq. (4) yields a nonzero remainder, indicating that his integral solution of Eq. (10) was incorrect, (details in Appendix).

3.3 Relationship between Defocusing and Focusing Solutions

The duality of Eq. (13), $y^+(t,x) = -y^-(t,x)$, reveals a structural correspondence between the defocusing and focusing branches. Because Eq. (1) is invariant under the simultaneous transformation y → −y and c → −c, every solution of



the defocusing equation is mapped to a solution of the focusing equation. The observed symmetry is therefore a direct consequence of the mathematical structure of the governing PDE.

3.4 Analytical Structure and Stability

The travelling-wave reduction of Section 2 reduces the KdV equation to a first-order ODE whose general solution is the Weierstrass elliptic function (Eq. (6)). The meromorphic structure of this solution is inherited directly from the PDE: the balance between the quadratic nonlinearity and the third-order dispersion fixes the order of the pole, so that each solution exhibits a double pole at every lattice point. When the elliptic invariants vanish, $g_2 = 0$ and $g_3 = 0$ (Eqs. (9) and (10)), the elliptic function degenerates into the rational soliton waves of Eqs. (11) and (12), which decay algebraically as $1/\xi^2$ as $|\xi| \to \infty$. For generic values of the invariants the solutions are smooth and bounded on the real axis and reproduce the classical periodic (cnoidal) waves of the KdV equation [1], whose discovery traces back to Boussinesq [18].

The KdV equation is a completely integrable Hamiltonian system [5] and possesses an infinite hierarchy of conserved quantities [2, 3]. Benjamin [19] established the orbital stability of the solitary waves in the sense of Liapunov: solutions that start close to a solitary wave remain close, up to translations and phase shifts, for all time. On the smooth periodic branch the same Hamiltonian structure [5] and the boundedness of the conserved quantities support the stability of the solutions; a detailed stability analysis of the periodic waves is beyond the scope of this paper.

The stability analysis is consistent with the discrete symmetry identified in Section 3.3. The governing equation is invariant under the simultaneous transformations $y \to -y$ and $c \to -c$; substituting $Y = -y$ and $C = -c$ into Eq. (1) leaves the equation invariant, so that the mapping $(y, c) \to (-y, -c)$ is an involution on the solution set. The reflection duality of Eq. (13) is therefore a direct consequence of the mathematical structure of the PDE rather than of any additional modelling assumption. This structural explanation, together with the exact cancellation of residuals in the symbolic verification of Section 3.2, demonstrates that the observed symmetry is an exact property of the model and is accessible directly from the equation itself.

4. Conclusion

We presented new exact analytical solutions of a nonlinear KdV equation, obtained via travelling-wave reduction and verified with symbolic computation. The results satisfy the expectations set out in the Introduction: they provide exact analytical benchmarks for nonlinear dynamics, and the duality between the focusing and defocusing regimes highlights the structural symmetry of the model. The analytical structure and stability of the solution branches were discussed in Section 3.4, and the Appendix corrects errors in previously published results. The methodology can be extended to other nonlinear systems, such as the Klein–Gordon equation, providing exact analytical benchmarks for theoretical and computational studies.

Appendix. Soliton Solutions of the Korteweg–de Vries Equation (Original vs Corrected)

The Korteweg–de Vries (KdV) equation is a fundamental nonlinear dispersive equation that arises in shallow-water wave theory, plasma physics, and the continuum limit of the Fermi–Pasta–Ulam–Tsingou (FPUT) lattice. Its standard normalized form is (Eq. (4) in [17]):

$$u_t + 6u \cdot u_x + u_{xxx} = 0$$

where u(x,t) denotes the wave elevation, and subscripts denote partial derivatives. The KdV equation admits localized travelling-wave solutions—solitons—arising from a balance between nonlinearity and dispersion.

A.1 Travelling-wave reduction

Let the solution travel to the right with constant speed $c > 0$:

$$u(x,t) = \ell(\xi)$$

$$\xi = x - c \cdot t$$

Then

$$u_t = -c \cdot \ell'$$



$$u_x = \ell'$$

$$u_{xxx} = \ell'''$$

Substituting into the KdV equation gives:

$$-c \cdot \ell' + 6\ell \cdot \ell' + \ell''' = 0$$

A.2 First integration

Integrating once with respect to ξ:

$$-c \cdot \ell + 3\ell^2 + \ell'' = C_1$$

For a solitary wave, ℓ, ℓ′, ℓ″ → 0 as |ξ| → ∞, so

$$C_1 = 0$$

Thus

$$\ell'' = c \cdot \ell - 3\ell^2$$

A.3 Second integration

Multiply by 2ℓ′ and integrate:

$$2\ell'\ell'' = 2(c\ell - 3\ell^2)\ell'$$

Since 2ℓ′ℓ″ = d(ℓ′²)/dξ, we obtain:

$$\ell'^2 = c\ell^2 - 2\ell^3 + C_2$$

Solitary-wave decay again requires:

$$C_2 = 0$$

Thus the reduced first-order equation is:

$$\ell'^2 = \ell^2(c - 2\ell)$$

Taking the positive branch:

$$\ell' = \ell\sqrt{(c - 2\ell)}$$

A.4 Correct separation of variables

$$\frac{d\ell}{\ell\sqrt{(c - 2\ell)}} = d\xi$$

A solution satisfying this relation is:

$$\ell(\xi) = \frac{c}{2} \cdot sech^2\left(\frac{\sqrt{c}}{2}(\xi - \xi_0)\right)$$

Therefore, the one-soliton solution of the KdV equation is:

$$u(x,t) = \frac{c}{2} \cdot sech^2\left(\frac{\sqrt{c}}{2}(x - c \cdot t - \xi_0)\right)$$

The parameter ξ₀ represents the phase shift.

A.5 Particular example

For c = 4 and ξ₀ = 0:

$$u(x,t) = 2 \cdot sech^2(x - 4t)$$



This soliton travels to the right with speed 4 and maintains its shape indefinitely.

A.6 Mathematical Errors in the Published IJEAP Version

Error 1 — Incorrect solution of first-order ODE (Equation (10))

Original text:

$$\ell(\xi) = \frac{c^2}{2} \cdot \left(\frac{\sqrt{c}}{2} (\xi - \xi_0) \right)$$

This is the incorrect integral of equation (9). Correct:

$$\ell(\xi) = \frac{c}{2} \cdot sech^2 \left(\frac{\sqrt{c}}{2} (\xi - \xi_0) \right)$$

Error 2 — Incorrect soliton solution (Equation (11))

Original text:

$$u(x,t) = \frac{c^2}{2} \cdot \left(\frac{\sqrt{c}}{2} (x - ct - x_0) \right)$$

Correct:

$$u(x,t) = \frac{c}{2} \cdot sech^2 \left(\frac{\sqrt{c}}{2} (x - c \cdot t - \xi_0) \right)$$

Error 3 — Incorrect particular solution (Equation (12))

Original text:

$$u(x,t) = 2^2 (x - 4t)$$

The scaling factor was not justified. Correct only because $\sqrt{c}/2 = 1$ when $c = 4$.

Correct:

$$u(x,t) = 2\, sech^2 (x - 4t)$$

Declarations

The author declares no competing interests, no data and no funding.

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