



Fractional Calculus Meets Mathematics Education

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Abstract

Fractional differential equations (FDEs) extend classical calculus to non-integer and even complex orders, providing powerful models for memory-dependent and nonlocal phenomena in science and engineering. At the same time, mathematics education increasingly relies on digital and AI-powered tools to support conceptual understanding, experimentation, and verification. This paper synthesizes three case studies on fractional calculus and the MathHandbook symbolic computation platform into an educational framework for teaching and learning fractional differential equations in secondary and tertiary settings, with a focus on Mathematics Education audience.

We review unified operator notation for integer, fractional, integral, complex, and variable orders; illustrate analytical solutions of ordinary and partial fractional differential equations; and highlight the role of validation tools such as symbolic and numerical testing. Building on the observation that “solution = general solution + particular solution = `gsolution()` + `psolution()`” and that particular solutions remain invariant across constant orders, we propose pedagogical strategies that leverage structural patterns, visualization, and AI-driven computation to scaffold students’ understanding of FDEs. The integration of MathHandbook into mathematics education demonstrates how students can explore fractional models, compare orders, and verify solutions interactively, thereby

bridging theory and computation and supporting inquiry-based learning in differential equations.

Keywords: fractional differential equations, fractional calculus, mathematics education, symbolic computation, AI in education, MathHandbook.

1. Introduction

Differential equations form the foundation of mathematical physics, modeling phenomena such as heat conduction, diffusion, elasticity, gravitation, and electrochemistry [1-6]. Classical models are built on integer-order derivatives, but many real-world systems exhibit memory, nonlocality, and hereditary behavior that are better captured by fractional differential equations. Fractional calculus [7-9] extends differentiation and integration to arbitrary real or complex orders, and FDEs now appear in physics, chemistry, biology, biophysics, economics, control theory, and signal processing [9-11].

At the same time, mathematics education is undergoing a transformation through online calculators, computer algebra systems, and AI-powered platforms. The online symbolic computation system MathHandbook (an evolution of the SymbMath computer algebra system) [12-15] provides a unified environment where students can input differential equations of arbitrary order, obtain analytical or numerical solutions, and validate them instantly. As one of the case studies notes, “The integration of AI into math education has opened new horizons for personalized, interactive, and effective learning. The AI tool empower students to conquer math challenges with confidence.”



For Mathematics Education, the central question is not only how fractional differential equations can be solved, but how they can be learned. This paper therefore focuses on the educational implications of fractional calculus: how unified notation, structural solution patterns, and interactive validation can be used to design learning experiences that make advanced topics accessible to diverse learners.

2. Unified Operator Notation for Fractional Differential Equations

2.1 The $ds(y, x, a)$ Operator

A key pedagogical feature of MathHandbook is the unified operator $ds(y, x, a)$, which represents derivatives and integrals of arbitrary order. This notation allows integer, fractional, negative, and complex orders to be expressed consistently, reducing notational barriers for students encountering fractional calculus for the first time. In the case study, the operator is defined piecewise to cover positive orders (derivatives), negative orders (integrals), and general real or complex orders. The operator is defined as:

$$\begin{aligned} ds(y, x, n) &= \frac{d^n y}{dx^n}, & n > 0, \\ ds(y, x, -n) &= \int^n y \, dx, & n > 0, \\ ds(y, x, \alpha) &= \frac{d^\alpha y}{dx^\alpha}, & \alpha \in \mathbb{R} \cup \mathbb{C}. \end{aligned}$$

For fractional calculus, MathHandbook uses the Caputo fractional derivative by default. For $0 < \alpha < 1$, the Caputo derivative is defined as:

$${}^c D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) \, dt.$$

When the user selects `lasolve`, MathHandbook instead applies the Riemann–Liouville definition:

$${}^{RL} D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^x (x-t)^{-\alpha} f(t) \, dt.$$

A key identity used throughout this study is the fractional derivative of the exponential function. Because e^x is an eigenfunction of both the Caputo and Riemann–Liouville operators, we have:

$$\frac{d^\alpha}{dx^\alpha} e^x = e^x, \quad \text{for all } \alpha \in \mathbb{R} \cup \mathbb{C}.$$

This property ensures that exponential terms appearing in general solutions retain their structure across all differential orders, which is essential for the unified analysis presented in later sections.

For example, the first-order linear differential equation

$$\frac{dy}{dx} - 2y = e^x$$

is entered as `dsolve( ds(y,x,1) - 2*y = exp(x) )`, while a half-order equation uses `ds(y,x,0.5)` and an integral equation uses `ds(y,x,-1)`. One of the summaries emphasizes that “Fractional derivatives and integrals [are] using expressions like `(y(0.5,x))`, `(ds(y,x,0.5))`, and `ints(y,x,0.5)`. It also covers: half-order derivatives and integrals, double-integral equations, PDEs in 1D, 2D, and 3D diffusion models.”

2.2 Table of Orders as a Learning Scaffold

The comparison tables of orders in the case studies provide a natural scaffold for learners. Table 1 in the fractional differential equation paper lists second-order, first-order, semi-differential, semi-integral, integral, double-integral, complex, and variable-order equations, all written in the same structural form $(\text{operator})y - 2y = \exp(x)$. This



uniformity allows students to see how changing the order modifies the operator while preserving the algebraic structure of the equation.

Table 1. Illustrative comparison of orders and operator notation

Order	Name	Equation (schematic)	Operator form
2	Second order	$\frac{d^2y}{dx^2} - 2y = e^x$	ds(y,x,2)
1	First order	$\frac{dy}{dx} - 2y = e^x$	ds(y,x,1)
0.5	Semi differential	$D_x^{0.5}y - 2y = e^x$	ds(y,x,0.5)
-0.5	Semi integral	$D_x^{-0.5}y - 2y = e^x$	ds(y,x,-0.5)
-1	Integral	$\int y dx - 2y = e^x$	ds(y,x,-1), ints(y,x)
i	Complex order	$D_x^i y - 2y = e^x$	ds(y,x,i)
cos(x)	Variable order	$D_x^{\cos(x)} y - 2y = e^x$	ds(y,x,cos(x))

From an educational perspective, such tables can be used in classroom activities where students predict solution behavior as the order varies, or classify equations by type (ODE, PDE, integral, mixed, complex, variable-order) based on the operator. This supports both conceptual understanding and symbolic fluency.

3. Analytical Solutions and Structural Patterns

3.1 General and Particular Solutions Across Orders

Table 2. Summary of solution structures of differential equations

Order	Equation	General Solution	Particular Solution
2	$\frac{d^2y}{dx^2} - 2y = e^x$	$C_1 e^{\sqrt{2}x}$	$-e^x$
1.5	$\frac{d^{1.5}y}{dx^{1.5}} - 2y = e^x$	$C_1 e^{\frac{2}{\sqrt{2}}x}$	$-e^x$
1	$\frac{dy}{dx} - 2y = e^x$	$C_1 e^{2x}$	$-e^x$
0.5	$\frac{d^{0.5}y}{dx^{0.5}} - 2y = e^x$	$C_1 e^{4x}$	$-e^x$
-0.5	$\frac{d^{-0.5}y}{dx^{-0.5}} - 2y = e^x$	$C_1 e^{x/4}$	$-e^x$
-1	$\int y dx - 2y = e^x$	$C_1 e^{x/2}$	$-e^x$
-2	$\iint y dx^2 - 2y = e^x$	$C_1 e^{x/\sqrt{2}}$	$-e^x$
i	$\frac{d^i y}{dx^i} - 2y = e^x$	$C_1 e^{\frac{i}{2}x}$	$-e^x$
cos(x)	$\frac{d^{\cos(x)} y}{dx^{\cos(x)}} - 2y = e^x$	$C_1 E_{\cos(x)}(2x^{\cos(x)})$	$-e^x$

One of the most striking findings in the case studies is the structural invariance of particular solutions across constant orders for the linear equation



$$D_x^\alpha y - 2y = e^x,$$

where α ranges over integer, fractional, and negative values. The comparison table of solutions shows that while the general solution changes with α , the particular solution remains $-e^x$ for all constant orders. The paper summarizes this as:

The table 2 shows that solution structures of differential equations are the same regardless its order:

$$\text{solution} = \text{general solution} + \text{particular solution} = \text{gsolution}(\alpha) + \text{psolution}(\alpha).$$

The above table shows that particular solutions of their differential equations are the same regardless with its order.

For teaching, this observation can be turned into a powerful concept-building exercise. Students can be asked to compute or inspect solutions for several orders (e.g., 2, 1.5, 1, 0.5, -0.5, -1, -2) using MathHandbook, then identify which parts of the solution depend on the order and which do not. This reinforces the idea of general versus particular solutions and highlights the robustness of forcing-term responses in linear systems.

3.2 Exponential and Mittag-Leffler Structures

The case studies also emphasize that exponential functions are eigenfunctions of both Caputo and Riemann–Liouville fractional derivatives, leading to exponential structures in general solutions across orders. For constant orders, the general solution typically has the form $C_1 e^{k(\alpha)x}$, where the growth rate $k(\alpha)$ depends smoothly on the order. For variable-order equations, the solutions involve Mittag-Leffler functions, which generalize exponentials and naturally encode memory effects.

For example, the variable-order equation

$$D_x^{\cos(x)} y - 2y = e^x$$

has a general solution of the form

$$y_g(x) = C_1 E_{\cos(x)}(2x \cos(x)),$$

where $E_{\cos(x)}$ denotes a Mittag-Leffler function. The paper notes that “When the order is variable, it is a variable order differential equation... Its solution is in Table 2, where $E_{\cos(x)}(x)$ is Mittag-Leffler function.”

Introducing Mittag-Leffler functions in an educational context can be challenging, but the variable-order example provides a concrete entry point: students can compare the exponential solutions for constant orders with the Mittag-Leffler solution for variable order, and discuss how memory and nonlocality are reflected in the functional form. Visualizing these solutions using MathHandbook’s plotting tools further supports intuition-building.

4. Fractional ODEs, PDEs, and Integral Equations in the Classroom

4.1 Ordinary Differential Equations and Numerical Visualization

MathHandbook supports both symbolic and numerical solutions of ordinary differential equations. Symbolic solutions are obtained via `dsolve`, while numerical solutions are visualized using `odeplot`. The summary of analytical methods explains that “`odeplot()` solves ODEs numerically and displays 2D or 3D plots. Users can choose first-order (y') or second-order (y'') forms. Interactive visualization (rotation, time adjustment) helps interpret solution behavior.”

In a learning environment, this dual capability allows instructors to design tasks where students first attempt to solve an ODE symbolically, then use numerical plots to explore solution behavior, stability, and sensitivity to initial conditions. For nonlinear equations that do not admit closed-form solutions, students can still engage with the dynamics through graphs, connecting qualitative analysis with computational experimentation.

4.2 Partial Differential Equations and Fractional Time

The fractional calculus papers extend classical PDEs—such as Laplace, Poisson, heat, and wave equations—to fractional time derivatives. For example, the heat conduction equation with fractional time order is written as



$$D_t^{0.5}y - a^2\Delta y = f(t, x),$$

where $0 < 0.5 < 1$ and Δ is the Laplace operator. The text explains that “When PDE is extended to the fractional time, the time order in PDE is changed to fractional time order... The heat conduction equation describes the physical laws of heat conduction and molecular diffusion.”

For students, fractional PDEs can be introduced as extensions of familiar models: starting from the classical heat equation, the time derivative order is gradually varied, and students observe how diffusion behavior changes. Using MathHandbook’s plot2D and plot3D functions, learners can visualize solutions in one, two, or three dimensions, and interactively adjust time to see memory effects and anomalous diffusion.

4.3 Integral and Fractional Integral Equations

Integral equations correspond to negative-order derivatives, and fractional integral equations extend this to non-integer negative orders. The case study describes how such equations can be converted to differential form by differentiation, solved, and then verified. For example, the semi-integral equation

$$D_x^{-0.5}y - 2y = e^x$$

is entered as $ds(y,x,-0.5) - 2y = \exp(x)$, and its solution appears in the same comparison table as the differential equations.

In teaching, this conversion process can be used to reinforce the relationship between differentiation and integration, and to show that fractional integral equations are not fundamentally different from fractional differential equations—they share solution structures and can be treated within a unified framework. This helps demystify fractional calculus for students who may initially see it as a completely new topic.

5. AI-Powered Validation and Inquiry-Based Learning

5.1 Symbolic and Numerical Testing of Solutions

A distinctive feature of MathHandbook is its built-in validation tools. Solutions obtained via dsolve can be checked symbolically using the “test” button or the test() function, and numerically using ntest() with random values. The summary of analytical methods states that “MathHand provides multiple validation tools: ‘test’ button: symbolic verification (checks if substitution gives zero). test() function: programmable testing with custom parameters. ntest(): numerical testing using random values for robustness. These tools ensure correctness and reliability of analytical solutions.”

From an educational standpoint, these tools support inquiry-based learning. Students can propose candidate solutions, input them into MathHandbook, and use test or ntest to verify whether the solutions satisfy the equation. This shifts the focus from passive reception of solutions to active exploration and conjecture testing, aligning with constructivist approaches in mathematics education.

5.2 Designing Learning Activities Around Fractional Models

The case studies suggest several types of activities that can be adapted for classroom use:

- **Order comparison tasks:** Students use MathHandbook to solve the same linear equation for different orders (e.g., 2, 1.5, 1, 0.5, -0.5, -1, -2), then analyze how the general solution changes and why the particular solution remains invariant.
- **Model extension tasks:** Starting from a classical PDE (heat or wave equation), learners extend the model to fractional time and explore the impact on solution behavior using 2D and 3D plots.
- **Mixed-order and system tasks:** Students investigate equations that combine fractional derivatives and integrals, or systems of fractional differential equations, to see how multiple orders interact and how solutions can be verified.



- **Complex and variable-order tasks:** Learners experiment with complex-order and variable-order equations, observing oscillatory-exponential and Mittag-Leffler behavior, and discussing potential applications in signal processing and anomalous diffusion.

These activities leverage the computational power of MathHandbook while keeping the focus on conceptual understanding, pattern recognition, and mathematical reasoning—core goals of science and mathematics education.

6. Implications for Mathematics Education

The synthesis of fractional calculus and AI-powered symbolic computation presented in these case studies has several implications for Mathematics Education:

- **Curriculum design:** Fractional differential equations can be introduced as extensions of existing topics in calculus and differential equations, using unified notation and structural patterns to maintain coherence. This supports vertical integration from secondary to tertiary levels.
- **Technology integration:** Platforms like MathHandbook demonstrate how online calculators and AI tools can be embedded into courses to provide instant feedback, visualization, and validation, aligning with contemporary calls for digital literacy and computational thinking in mathematics education.
- **Equity and accessibility:** By lowering the technical barriers to exploring advanced topics, AI-powered tools can broaden participation in mathematics, allowing students from diverse backgrounds to engage with fractional models and differential equations that were previously reserved for specialized courses.
- **Research opportunities:** The structural properties of fractional solutions—such as the invariance of particular solutions and the emergence of Mittag-Leffler functions—invite further educational research on how students perceive and understand these patterns, and how they transfer knowledge between integer and fractional contexts.

As one conclusion in the case studies puts it, “We hope that the examples and ideas outlined in this paper will be useful for elementary and advanced courses on differential equations, as well as for solving differential equations which occur in research and design problems in practice.” Extending this hope to the broader community of science and mathematics educators, fractional differential equations and AI-powered symbolic computation offer fertile ground for innovation in teaching, learning, and assessment.

7. Conclusion

This paper has combined three complementary case studies on fractional differential equations and the MathHandbook platform into an educational narrative aimed at Mathematics Education. By highlighting unified operator notation, structural solution patterns, fractional ODEs and PDEs, integral and mixed-order equations, and AI-powered validation tools, we have argued that fractional calculus can be made accessible and engaging for students through thoughtful integration of technology and pedagogy.

The key message is that learning fractional differential equations need not be an abstract or purely theoretical endeavor. With platforms like MathHandbook, students can experiment with orders, visualize solutions, verify their conjectures, and connect mathematical structures to physical phenomena. As technology evolves, such tools will play an increasingly central role in personalized and exploratory learning, helping to realize a future where every student can explore advanced mathematical ideas—including fractional calculus—with confidence and curiosity.

Declarations

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References

1. W. Huang and D. B. Hibbert, Phys. Rev. E, 52(2), 5065–5069 (1995).
2. W. Huang and D. B. Hibbert, Phys. Rev. E, 53(1), 727–730 (1996).
3. J. Jiang, W. Huang and B. Hibbert, Physica A, 233(3–4), 884–887 (1996).
4. W. Huang and D. B. Hibbert, Physica A, 233(3–4), 888–896 (1996).
5. W. Huang, Europhysics Letters, 154(4), 42002 (2026), DOI: 10.1209/0295-5075/ae647a.
6. W. Huang, J. Research in Applied Mathematics, 12(2), 66–69 (2026).
7. K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, London (1974).
8. B. Ross, *Fractional Calculus and Its Applications*, Springer, Berlin, Heidelberg (1975).
9. K. B. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).
10. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York (1999).
11. Y. Hu, Y. Luo, Z. Lu, “Analytical solution of the linear fractional differential equation by Adomian decomposition method,” *Applied Mathematics and Computation*, 215(1), 220–229 (2008).
12. W. Huang, “SymbMath: A Program for Symbolic Mathematics,” *Int. J. Math. Edu. Sci. Technol.*, 23(1), 160–165 (1992).
13. W. Huang, “SymbMath Update to Version 2.0,” *Abstracts Amer. Math. Soc.*, 13(6), 535 (1992).
14. W. Huang, J. Software Engineering and Simulation, 12(2), 46–51 (2026).
15. W. Huang, International J. Research and Scientific Innovation, 13(7), 5392-5396 (2026).
<https://doi.org/10.51244/IJRSI.2026.1307000397>