



Fractional-Order Advection–Diffusion Equations: A Critical Review of Physical Origins, Analytical and Numerical Methods, and Applications

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Abstract

The advection–diffusion equation is the canonical continuum description of transport, yet it fails for a broad class of systems in which particle motion is non-Fickian. Fractional-order advection–diffusion equations replace integer derivatives with operators of non-integer order and, in doing so, encode long-range spatial jumps and long-memory waiting times that classical models discard. This review provides a critical, evidence-based synthesis of the field from its foundations in continuous-time random walks and Lévy motion through the mathematical and numerical landscape. We first examine the family of fractional operators and the physical meaning of their kernels. We then trace the micro–macro derivations that connect fractional advection–diffusion equations to anomalous transport, mobile–immobile systems, tempered and variable-order processes. Analytical methods and the now-mature toolbox of finite difference, finite element, spectral, meshless and lattice-Boltzmann discretizations are appraised for accuracy, stability and computational cost. Inverse problems and parameter estimation, which decide whether such models earn their extra complexity, are reviewed next. Applications in hydrology, porous media, plasmas, space physics, heat and mass transfer, biology and finance illustrate both success and unresolved tension. Throughout, we identify trends—rising use of non-singular-kernel operators, machine-learning solvers and realistic benchmarking—and we flag persistent gaps: inconsistent operator conventions, sparse validation against real data, and scarce higher-dimensional and multiscale results. We close with a concrete research agenda for making fractional advection–diffusion models predictive rather than descriptive.

Keywords: fractional calculus; anomalous diffusion; advection–diffusion equation; Lévy motion; Caputo derivative; mobile–immobile transport; finite difference methods; inverse problems.

1. Introduction

Transport of matter, energy and information by the combined action of drift and spreading is ubiquitous. Its workhorse model, the advection–diffusion (or advection–dispersion) equation (ADE), balances a first-order spatial derivative describing bulk motion against a second-order derivative describing random spreading. For a scalar concentration $u(x, t)$ in one dimension it reads

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + D \frac{\partial^2 u}{\partial x^2},$$

with drift velocity v and diffusivity D . This parabolic equation, and its Fickian assumptions of local flux and Gaussian spreading, underpin models of groundwater contamination, atmospheric dispersion, heat transfer, plasma transport and biological motion [1,2].

A century of tracer experiments tells a different story. Plumes in heterogeneous aquifers spread faster than linear, retain heavy tails, and produce breakthrough curves that rise and decay as power laws rather than exponentials [3-5]. Particles in turbulent plasmas perform long excursions [6,7], and tracers in rivers and soils display memory of their



history [8,9]. Such observations are the signature of anomalous, non-Fickian transport, and they have been catalogued in comprehensive reviews of fractional dynamics [10]. The classical ADE cannot reproduce them: its Green function is a Gaussian that spreads like $t^{1/2}$ and decays exponentially in space [2].

Fractional calculus offers a compact remedy [11-13]. If the waiting-time or jump-length statistics of a random walk are power-law distributed, the transport equation that emerges in the diffusion limit contains fractional rather than integer derivatives [2,14,15]. The resulting fractional advection–diffusion equation (fADE) reproduces heavy-tailed plumes, non-linear spreading rates and non-local flux in a single, parameter-sparse model [1,16,17]. Its fundamental solutions are Lévy-stable densities, which naturally accommodate skewness and power-law decay [16].

The field has grown explosively. Space-fractional, time-fractional and space-time-fractional forms have been proposed [1,18,19]; tempered and variable-order generalizations extend their reach [20-22]; and an ever-larger numerical literature now provides convergent and stable discretizations [23-25]. Recent methodological reviews [26] and operator comparisons [27,28] reflect a maturing but still contested discipline. New operators with non-singular kernels [29,30] have generated both excitement and sharp criticism [27], and by 2026 machine-learning solvers and data-driven order estimation have entered the mainstream [31,32].

This review offers a balanced, critical appraisal rather than an encyclopedic listing. We evaluate published evidence, identify what is established and what remains speculative, and highlight concrete research directions. The scope is deliberately broad—physical derivation, functional operators, analysis, analytic and numerical solution techniques, inverse problems and applications—because the field's fragmentation across communities is itself part of the problem. The literature is traced through 2026, with emphasis on work that either changed practice or exposed weakness. We aim to help practitioners select defensible models and methods, and to help theorists target questions that matter.

2. Fractional Derivatives and the Operator Zoo

Fractional derivatives are not unique. Different definitions agree on polynomials and on the exponential, but differ on general functions, boundary behavior and kernel. The choice is not cosmetic: it determines whether initial conditions are physical, whether the operator is local or non-local, and whether the equation inherits conservation or maximum principles [13,33]. The three classical families dominate practice.

The Riemann–Liouville (RL) fractional derivative of order $\alpha \in (n - 1, n)$ acts as

$$({}^{RL}D_x^\alpha u)(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(\xi)}{(x - \xi)^{\alpha - n + 1}} d\xi,$$

a Volterra-type operator whose power-law kernel makes it non-local in space [11,12]. Differentiating a fractional integral of order $n - \alpha$, it is the natural generalization of repeated integration. The Caputo derivative instead differentiates first,

$$({}^CD_x^\alpha u)(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{u^{(n)}(\xi)}{(x - \xi)^{\alpha - n + 1}} d\xi,$$

so that integer-order initial and boundary conditions appear naturally in initial-value problems [13,33]. For this reason the Caputo form dominates time-fractional modelling [26]. The Grünwald–Letnikov (GL) definition, constructed directly as the limit of fractional differences, provides the discrete foundation on which most finite-difference methods are built [23,24,34]. The three are equivalent for sufficiently smooth functions under suitable boundary conditions, an equivalence that underpins the interchange of analytical and numerical treatments [12,34]. Parameterized variants such as the k-Riemann–Liouville derivative adjust the operator for systems with scale-invariant kernels [35].

The classical operators carry singular power-law kernels. To model relaxation and memory without singular kernels, Caputo and Fabrizio proposed an exponential kernel and Atangana and Baleanu a Mittag-Leffler kernel [29,30,36], while the conformable derivative of Khalil et al. and Abdeljawad returned to a limit-of-differences definition that preserves many classical rules of calculus [29,30]. These proposals revitalized the field, particularly in biology and image processing [37-39], but they also triggered sustained critique. Angstmann et al. showed that solutions of initial-value problems involving Caputo–Fabrizio and Atangana–Baleanu operators generally have an intrinsic discontinuity



at the origin, an artifact that is hard to reconcile with physical modelling unless fractional initial conditions are accepted [27]. Because the physical meaning of the kernels and the correct initial conditions remain debated, we echo the growing consensus that operator choice must be justified by the microscopic model, not by algebraic convenience [27,28].

Two further constructions matter for transport. The Riesz derivative, a symmetric combination of left and right RL derivatives, generalizes the fractional Laplacian and is the canonical spatial operator for isotropic superdiffusion [19,25,28]. Variable-order and distributed-order operators, in which the exponent becomes a function of space, time or both, or is smeared over a weight distribution, capture media whose heterogeneity changes the local memory structure [22,40,41]. Distributed-order equations are the scaling limits of random walks with a spectrum of waiting-time exponents [41]. This zoo of operators, summarized in Table 1, is both the field's richness and its liability: results obtained with one definition rarely transfer to another without re-derivation.

Table 1. Representative fractional operators and their role in advection–diffusion modelling.

Operator family	Kernel / structure	Primary use in transport	Non-local
Riemann–Liouville	Power law (Volterra)	Space-fractional flux, whole-space models [1,11,12]	Yes (space)
Caputo	Power law, derivative last	Time-fractional memory, physical initial conditions [13,26,64]	Yes (time)
Grünwald–Letnikov	Fractional differences	Discretization basis for finite-difference schemes [23,34]	Yes (space)
Riesz	Symmetric one-sided RL pair	Isotropic superdiffusion, fractional Laplacian [19,25]	Yes (space)
Conformable	Limit of differences	Classical-rule local generalisation [29,30]	No
Caputo–Fabrizio	Exponential, no singularity	Weakly non-local relaxation [42]	Weak (kernel)
Atangana–Baleanu	Mittag-Leffler kernel	Relaxation, epidemiology models [36,39]	Weak (kernel)
Variable / distributed order	Order varies or is weighted	Heterogeneous and multi-scale media [22,40,41]	Yes

3. Deriving Fractional Advection–Diffusion Models

Fractional advection–diffusion equations are best understood as scaling limits of random walks, not as ad hoc replacements of derivatives. In a continuous-time random walk (CTRW), particles wait a random time and then jump a random length [2,14,15]. The CTRW picture also accommodates reaction-diffusion systems, whose normal and anomalous regimes differ qualitatively [43]. If the jump-length density has infinite variance (Lévy flights with index $0 < \alpha < 2$), the master equation converges in the long-time, large-scale limit to a space-fractional Fokker-Planck equation whose spatial derivatives are fractional of order α [2,16,44]. Benson, Wheatcraft and Meerschaert established this connection for solute transport and showed that the resulting fractional ADE has Lévy-stable fundamental solutions that spread like $t^{1/\alpha}$, carry heavy tails and allow arbitrary skewness [1,16]. An Eulerian derivation, framed through fractional conservation of mass, made the model physically transparent and identified the fractional divergence as the correct generalization of the flux [17,45]. The one-dimensional space-fractional ADE takes the form



$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + D[p {}^{RL}_{-\infty}D_x^\alpha + (1-p) {}^{RL}_xD_\infty^\alpha]u,$$

where p weights upstream and downstream jumps and the pair of one-sided RL derivatives encodes direction-dependent non-locality [1,24]. Two-sided fractional dispersion equations generalize this to multi-dimensional anisotropic media [45,46]. Textbooks and book chapters now frame the fractional advection-dispersion equation as the standard model for contaminant transport in geophysics [47].

When the waiting-time distribution instead has infinite mean (index $0 < \gamma < 1$), the CTRW limit is a time-fractional equation. The time-fractional ADE

$$({}_0^CD_t^\gamma u) = -v \frac{\partial u}{\partial x} + D \frac{\partial^2 u}{\partial x^2}$$

describes subdiffusion with drift, in which the mean-squared displacement grows like t^γ and the process is non-Markovian [18,48]. Its connection to CTRW with Mittag-Leffler waiting times is classical [2,14,15]. Nonlinear fractional Fokker–Planck equations, motivated by nonextensive statistical mechanics, extend these descriptions to interacting and porous-type media [49]. Schumer, Benson, Meerschaert and Baeumer generalized the concept to fractal mobile/immobile (MIM) transport, in which a fraction of mass exchanges with an immobile phase whose residence times are power-law distributed; the result couples an integer time derivative for the mobile phase to a fractional derivative for the immobile phase [50]. The MIM model and its fractional variants reproduce power-law breakthrough curves at field sites where classical models fail [50,51]. Subordination offers an alternative route: Baeumer et al. showed that applying a random time change (subordination) to a classical solution produces the same heavy-tailed phenomenology without adding new spatial operators [4].

Combining space and time fractions yields the space-time-fractional ADE, whose governing operator interpolates between memory in time and non-locality in space [18,19,25]. Povstenko and collaborators systematically derived such equations from two alternative physical premises—replacing the time derivative in the classical balance, versus replacing the spatial flux—and showed the two routes give different equations and different fundamental solutions [19,52]. Multiscaling fractional ADEs, in which different spatial directions carry different fractional orders (even matrix orders along non-orthogonal fracture orientations), extend the family to anisotropic media and fracture networks [51]. Tempered fractional models multiply the power-law jump kernel by an exponential cutoff, restoring finite moments and physical transport with bounded velocities while preserving the desirable small-scale scaling [20,21]; moments of their solutions exhibit a transition from anomalous to near-Fickian behavior at long times [21]. Variable-order versions let the exponent vary with location or state, capturing media whose heterogeneity changes locally [22,40], and distributed-order models integrate over a continuum of exponents to reproduce a spectrum of memory scales [22,41]. Recently, generalized kernels beyond the pure power law—including Sonin-kernel and Mittag-Leffler structures—have been proposed as more flexible anomalous-diffusion models [53].

Several caveats frame this section. First, different derivations of “the” space-time fractional ADE are not equivalent [19], so equations should be reported with their derivation. Second, the MIM and subordination viewpoints often fit the same data, and discriminating among them requires independent physical information [4,50]. Third, adding fractional structure increases parameter count; Occam’s razor argues for tempered or variable-order forms only when data demand them [21,40].

4. Mathematical Analysis and Well-Posedness

Before computing, one must know that a solution exists, is unique and depends continuously on data. Analysis of fractional advection–diffusion equations is less complete than for classical parabolic problems, but the picture is stabilizing. McLean, Mustapha, Ali and Knio established well-posedness for time-fractional advection–diffusion–reaction equations, providing the regularity theory that numerical error analyses rely upon [54]. Kemppainen proved existence and uniqueness for time-fractional diffusion equations on bounded domains [55]. Silvestre derived Hölder estimates for advection problems with fractional diffusion, showing that fractional diffusion regularizes even in the presence of strong drift [56]. For the space-fractional ADE, existence and uniqueness of solutions representable as



fractional power series was established by Dimache et al., with a constructive approximation scheme and error control [57].

The questions that remain open are structural. Fractional operators do not, in general, preserve the strong maximum principle that guarantees positivity of concentrations, and sign-preservation results are scarce for two-sided and space-time-fractional problems [52,54]. Boundary conditions acquire a new subtlety: for one-sided spatial fractional operators, the correct boundary data are not simply point values at the endpoints but fractional or trace conditions that depend on the operator's orientation, and errors in setting them propagate through the whole domain [8]. Povstenko analyzed generalized (non-perfect) contact conditions for time-fractional advection-diffusion, an important but rarely treated topic for layered media [52,58]. Space-time fractional boundary value problems have also been solved by separation of variables, with eigenfunctions that reveal the interplay of drift and fractional diffusion [59]. For conformable and other non-standard operators, the question of whether initial conditions carry the classical physical meaning is itself unsettled [27,30]. Higher-dimensional regularity, boundary-layer behavior at the singular origin for time-fractional problems, and nonlinear reaction couplings remain active research fronts [60,61]. Where engineering design is concerned, optimal control of time-fractional advection-diffusion systems has begun to receive attention [62].

5. Analytical Solution Methods

Closed-form and semi-analytical solutions remain essential for benchmarking numerical schemes and for physical insight. Because fractional operators are linear and translation-invariant on whole space, integral transforms are the natural tool. The Caputo time-fractional equation is amenable to Laplace transforms combined with Fourier or Hankel transforms in space, and the solutions are expressed through Mittag-Leffler and Fox H-functions [15,18,42]. Gill, Singh and Singh coupled Sumudu and Fourier transforms to solve a generalized space-time fractional ADE in closed form [63]. Povstenko and Kyrylych used Laplace–Fourier pairs to construct fundamental solutions of the space-time fractional ADE [19], and Stern et al. derived analytical solutions for the space-fractional diffusion–advection equation and used them to expose systematic errors in common numerical approximations as the fractional order approaches unity [28]. Rubbab, Mirza and Qureshi obtained analytical solutions for the time-fractional ADE with Caputo–Fabrizio kernels under time-dependent boundary pulses, showing that memory delays the solute front [42]. Analytical time-dependent solutions for finite-speed electrodeposition transport, which couple diffusion, convection, drift and relaxation, were recently obtained via eigenfunction expansion, providing testable predictions for transient electrochemical response [64]. A two-parameter Mittag-Leffler representation of the space-fractional advection-diffusion equation similarly describes sediment suspension profiles in turbulent open-channel flows [65]. Dirichlet problems for the time-fractional advection-diffusion equation on line segments have been solved by integral-transform methods, connecting the Caputo operator to classical potential theory [66].

For nonlinear and variable-coefficient problems, a family of approximate-analytical methods is widely used. Exact time-dependent solutions for nonlinear fractional Fokker–Planck-like equations are rare but valuable reference cases [67]. The homotopy perturbation method was applied to the space-time fractional ADE by Yildirim and Koçak [68] and extended to space- and time-fractional forms by Singh and Chatterjee [69]. The homotopy analysis method, which contains an auxiliary convergence-control parameter, was used for the one-dimensional time-fractional ADE by Singh, Chatterjee and Singh [70]. Adomian-type decompositions and their Laplace variants have been adapted to time-fractional advection-dispersion equations, and Legendre collocation gives spectral accuracy for the generalized fractional ADE [71]. Mittag-Leffler-kernel formulations underpin several efficient approaches for time-fractional advection–reaction–diffusion systems [72], and Rabotnov fractional-exponential kernels have been used for space-time fractional advection-reaction-diffusion [73]. Double-integration techniques recast nonsingular-kernel fractional advection-dispersion problems into forms amenable to quadrature [74]. These series methods are easy to implement and often strikingly accurate on benchmark problems, but they lack robust a priori error control for nonsmooth data, and their convergence domains depend on problem parameters; we recommend treating them as tools for benchmarks and for qualitative insight rather than as black-box solvers for production transport codes [68,70].



6. Numerical Methods

Numerical solution is where fractional advection–diffusion modelling meets practice. Fractional operators are non-local, so naive discretization yields dense matrices, memory-heavy history sums and subtle stability constraints. The literature has responded with increasingly sophisticated schemes, benchmarked against manufactured solutions and the analytic results of Section 5 [28,57].

6.1 Finite difference methods

The workhorse is the shifted Grünwald–Letnikov discretization of one-sided RL derivatives, which Meerschaert and Tadjeran stabilized for two-sided space-fractional equations by shifting the finite-difference stencil [23,24]. Their implicit schemes for the space-fractional ADE are unconditionally stable and convergent [23], and the framework extends to two-dimensional fractional dispersion [46]. Shen, Liu and Anh developed numerical approximations for the space-time Riesz–Caputo fractional ADE with rigorous stability and convergence analysis [25]. Li and Deng designed a family of difference schemes with improved accuracy for space-fractional advection–diffusion [75], and Abdel-Rehim proposed an implicit scheme for the space-time fractional ADE and analyzed its consistency [76]. Nonstandard finite-difference schemes preserve qualitative features of two-sided fractional problems that standard schemes distort [77]. For time-fractional subdiffusion, L1-type discretizations of the Caputo derivative are standard, with graded meshes restoring optimal convergence for weakly singular solutions [25,54]. Because the history term couples every time level to all previous ones, fast and parallel solvers matter; Crank–Nicolson discretizations combined with Krylov iterative solvers and explicit group methods have demonstrated large speedups for variable-order problems in two dimensions [78], and efficient treatments of variable-order time-fractional advection–diffusion now report robust accuracy on coarse meshes [79].

The central difficulty of finite-difference methods is memory: the GL sum and the L1 history sum grow with every time step, making naive implementations $\mathcal{O}(N_t^2)$ in time [25,78]. Efficient algorithms, including exponential-sum approximation of kernels and parallel-in-time ideas, are becoming standard [78,80]. Implicit difference schemes built on variable-order Riemann–Liouville integrals and conformable-type formulations have likewise been derived and analyzed for variable-order sub-diffusion and nonlinear reaction–advection problems [81,82]. Stability analyses for the non-standard operators (Caputo–Fabrizio, Atangana–Baleanu, variable-order) are less complete, and several published schemes converge only under restrictive smoothness assumptions on the solution [28,54].

6.2 Spectral, collocation and wavelet methods

Spectral methods achieve exponential convergence when solutions are smooth. Chen et al. derived optimal error estimates for Legendre spectral approximation of the space-fractional reaction–advection–diffusion equation [83]. Zaky’s finite-difference/Petrov–Galerkin spectral framework handles time-space fractional reaction–diffusion with non-smooth data by combining graded L1 time stepping with generalized basis functions [84]. Jacobi spectral collocation solves variable-order fractional mobile–immobile advection–dispersion models arising in solute transport [85], and Chebyshev spectral methods have been applied to the same class of problems with variable order and complex media [86]. Wavelet bases—fractional Chelyshkov wavelets for distributed-order MIM advection–dispersion [87], fractional Taylor wavelets for variable-order diffusion [88], and hybrid Chebyshev-cardinal/modified-Legendre bases on unbounded domains [89]—offer sparse representations and adaptive refinement. A substantial meshless literature uses radial basis functions (RBF) for fractional advection–diffusion and subdiffusion, both for smooth problems [90,91] and for stochastic fractional advection–diffusion equations where classical methods struggle [92], with local RBF-finite-difference upwinding handling distributed-order time-fractional advection–diffusion [93]. Laplace-transform/local-RBF combinations similarly treat fractional mobile-immobile solute transport without time-stepping instabilities [94]. Variable-order formulations of the reaction–diffusion and mobile-immobile classes have been solved with Chebyshev and Legendre-type collocation and with Mittag-Leffler operators [95,96]. These high-order methods are accurate but sensitive to regularity; for space-fractional problems the solution’s smoothness is typically limited by the fractional order and the boundary treatment, so unfounded smoothness assumptions silently degrade convergence [28,84].



6.3 Finite element and other discretizations

Finite element methods give variational structure and mesh flexibility. Zheng, Li and Zhao analyzed the finite element method for the space-fractional ADE and clarified the correct weak formulation and its consequences for error estimates [97]. Dehghan and Abbaszadeh combined finite difference and finite element techniques for the space-fractional tempered diffusion-wave equation with explicit error estimates [98]. Galerkin and weighted-residual formulations continue to advance: Bin Jebreen and Wang solved the space-fractional ADE by a Galerkin method on Chebyshev-cardinal bases with convergence proof [99], and a 2026 study modeled and solved space-fractional advection-diffusion problems by the Galerkin weighted-residual method with Bernoulli-polynomial bases, reporting high accuracy and easy implementation [100]. Spectral collocation with Fibonacci and other orthogonal polynomials has been applied to nonlinear space-time fractional advection-reaction-diffusion transport [101]. Space-time finite element error analyses for classical advection–diffusion continue to inform the fractional case [102].

Lattice Boltzmann methods (LBM) provide a mesoscopic alternative. Cartalade, Younsi and Néel developed a multiple-relaxation-time LBM for the fractional ADE [103], and a 2025 LBM for tempered space-fractional advection–diffusion recovers the target equation to second order and simulates tempered transport from a point source [104]. Upwind space-time discretizations of fractional advection–dispersion in fractured systems have also been analyzed [105]. For two-dimensional problems, the combination of finite differences with meshless RBF local collocation remains among the most flexible and least implemented options [90,92,106].

7. Parameter Identification and Inverse Problems

A fractional model is only as credible as its estimated parameters, and the order α or γ is the hardest to identify. When the exponent can be recovered from data, it is often treated as an effective index of heterogeneity or of medium structure rather than a microscopic constant [107,108]. Tikhonov-regularized identification of dispersion and order parameters for the space-fractional ADE from solute-breakthrough data has been demonstrated [108,109]. Bees-algorithm optimization successfully recovered fractional transport parameters in soil-column experiments [108], and classical results show that the fractional order leaves distinctive fingerprints in time traces and final measurements that permit stable recovery [107,110]. Janno and Kinash proved uniqueness and stability for simultaneous recovery of the fractional order and a space-dependent source from final-time data [110]; Rundell and Zhang established uniqueness for recovering distributed (multi-order) fractional derivatives from overposed data [107]. For distributed-order time-fractional diffusion, Sun and Liu solved the inverse source problem with conditional stability and a conjugate-gradient scheme [111]. Variable-order recovery is harder still; Artyushin studied the inverse problem of reconstructing a space-dependent order from boundary data [112], and a 2026 study established well-posedness for inverse source problems in variable-order time-fractional diffusion [32]. Data-driven machine-learning approaches, including fractional physics-informed neural networks, now offer a practical route to estimate distributed-order structure from noisy and sparse observations [31].

The recurring message is that fractional-order estimation is ill-posed: small noise can move the estimated order appreciably, orders are weakly identifiable when data are sparse or cover a narrow time window, and different model families (space- vs time-fractional) can produce similar fits to the same plume [110,113]. Reliable practice therefore requires regularization, careful experimental design and, ideally, multiple independent observables [108,111].

8. Applications

8.1 Hydrology, porous media and environmental transport

The fractional ADE was born in hydrology. Benson, Wheatcraft and Meerschaert's analysis of laboratory and field tracers—including the Cape Cod bromide plume—showed that fractional orders near 1.5–1.8 reproduce heavy-tailed plume shapes that classical models cannot [1,3]. Field evidence for scale-dependent fractional advection–dispersion consolidated the approach [5], and fractional MIM transport described the anomalous mass decline at the MADE site [50]. Eulerian and subordination derivations placed these models on firm footing [4,17], while fractional vector calculus extended them to arbitrary geometries [45]. Later work addressed boundary effects in soil, showing that the choice of Caputo-type dispersive flux is decisive near interfaces [8], and upscaling studies demonstrated that time-



fractional ADEs reproduce solute transport in rough single fractures with matrix diffusion [9]. Nonlinear reaction-advection-diffusion fractional models have been studied for porous media transport [114]. Recent comparisons still find that space-fractional models outperform classical ones in heterogeneous soils [115], and parameter-estimation studies in real aquifers continue to refine practice [108,109].

8.2 Plasmas, space physics and turbulence

Plasma transport provided some of the cleanest evidence for nonlocality. del-Castillo-Negrete, Carreras and Lynch showed that tracer transport in resistive pressure-gradient-driven plasma turbulence is superdiffusive and quantitatively reproduced by a space-time fractional model [6,116]. Fractional diffusion-advection equations now describe energetic-particle transport in the heliosphere [7,117], where Lévy-flight models reproduce power-law particle profiles upstream of shocks [117], and in solar-wind turbulence more generally [118]. These applications are among the few in which fractional models have been validated against particle statistics inferred from observations rather than only against plume fits [6,117].

8.3 Heat and mass transfer

Fractional formulations of heat conduction replace Fourier's law with memory-type flux laws, producing time-fractional and Cattaneo-type models with finite propagation speed. Analytical solutions for time-fractional Cattaneo heat conduction under pulsed flux were derived by Xu and Wang [119]. Fractional Cattaneo heat-flux models have been embedded in nanofluid and Maxwell-fluid convection problems [120,121], and anomalous heat diffusion has been connected to the fractional Fokker-Planck equation [122]. These studies demonstrate that fractional models can rationalize non-Fourier heat transfer that classical and even hyperbolic (Cattaneo) models misrepresent [119,122]. The electrochemical community has begun to use finite-speed, memory-type transport descriptions for electrodeposition [64].

8.4 Biology, medicine and beyond

Fractional reaction-diffusion systems model superdiffusive and memory-driven transport in excitable tissue: a fractional-order model of neuronal calcium and inositol-trisphosphate dynamics showed how superdiffusion and memory dysregulate signaling in disease [123]. In epidemiology and physiology, non-singular-kernel operators are now pervasive in compartment and transport models, including a fractional-order model of Ebola virus transmission [37] and fractional Keller–Segel descriptions of chemotaxis [39], though several of these studies inherit the operator-choice caveats of Section 2 [27]. In imaging, Caputo–Fabrizio fractional masks sharpen edge detection beyond classical integer masks [38]. In finance, time-fractional diffusion has entered option-pricing models, where spectrally accurate solvers for the time-fractional Black–Scholes equation were recently demonstrated [124]. Air-pollution modelling with time-fractional advection–diffusion equations is an emerging application [125].

8.5 Critical reading of the application literature

The application literature is uneven. The most convincing studies pair fractional models with independent physical motivation and real data—field plumes [1,50], turbulence statistics [6,116], microchannel experiments [5]. Weaker studies fit a fractional order to synthetic data and conclude the model “captures” the phenomenon without evidence of predictive gain [26,28]. Reviewers should demand: (i) a microscopic or statistical justification for the operator; (ii) identification of which observed anomaly the model explains; and (iii) a comparison with the best classical alternative on held-out data [28,110].

9. Critical Assessment and Open Debates

Four debates structure the current literature.

First, the meaning of the order. In space, α indexes the heaviness of jump tails; in time, γ indexes memory. Yet many papers estimate them as free parameters and reinterpret them post hoc [107,108]. The field needs accepted standards linking order to measurable medium properties [5,112].



Second, operator proliferation. Dozens of “new” fractional derivatives—conformable, exponential, Mittag-Leffler, Sonin-kernel, variable, distributed—appear each year. Only those with a clear microscopic origin and consistent initial conditions have proven durable [27,29,30]. The mathematical community has repeatedly shown that non-singular-kernel operators with altered initial conditions are not interchangeable with classical fractional derivatives [27]; reviews increasingly call for operator parsimony [26,28].

Third, numerics versus reality. Finite-difference and spectral methods are mature in one dimension, but higher-dimensional, heterogeneous and nonlinear problems remain hard; history dependence makes time stepping expensive, and accurate boundary treatment for two-sided operators is delicate [8,46,78]. Many published convergence proofs assume more regularity than physical solutions possess [28,54].

Fourth, validation culture. Analytic solutions exist for a good range of linear problems [18,28], yet many papers validate only against manufactured or exact solutions of the same equation—an internal check, not external evidence. The most persuasive work confronts field or experimental data and compares model classes [3,110,116]. Establishing shared benchmark problems with real datasets is the single most effective step the community could take [28,32].

10. Emerging Trends and Future Directions (2024–2026)

Several trends define the current period. Machine-learning solvers are the most visible. Physics-informed neural networks (PINNs) that embed fractional residuals in the loss are maturing: adaptive fractional-polynomial PINNs now solve distributed-order time-fractional diffusion with error analysis and randomized residual refinement [31]. Fractional physics-informed approaches are also being used for inverse problems in which the order itself is learned from data [32]. We expect hybrid solvers—finite-difference or spectral discretizations with neural corrections—to become standard for high-dimensional and stochastic fractional transport [31,90].

Generalized and physically grounded kernels are a second trend. Sonin-kernel and tempered generalized fractional diffusion models now provide more flexible memory structures while preserving the probability interpretation of the solution [21,53]. Careful operator analysis is replacing the earlier enthusiasm for ad hoc kernels [27,36].

Third, scalable algorithms. Variable-order, tempered and distributed-order problems in realistic geometries demand fast convolution, parallel time stepping and mesh adaptivity; explicit-group Krylov methods and graded-mesh techniques are representative advances [78,80,84].

Fourth, a push toward standard benchmarks and reproducibility. Recent reviews emphasize re-analysis, operator comparisons and reproducible code [26], and inverse-problem papers increasingly test on noisy, sparse and non-ideal data [32,112].

Looking ahead, we identify five priorities. (1) Establish physically motivated derivation chains for every model family and publish them with equations, so “the fractional ADE” ceases to be ambiguous [19]. (2) Build community benchmark suites from real hydrological, plasma and heat-conduction datasets with fixed reporting protocols [3,6]. (3) Develop rigorous boundary and interface conditions for two-sided and space-time operators in complex domains [8,52]. (4) Strengthen higher-dimensional and multiscale numerical analysis, including stochastic forcing [46,92]. (5) Create software that implements several operators and solvers side by side, so that model selection is evidence-based rather than habit-based [26,28]. In parallel, machine-learning and physics-informed methods should be directed at the hardest part of the problem—order and memory identification from partial, noisy data—where their interpolation power is most valuable and their danger of overfitting is best controlled by fractional priors [31,32].

11. Conclusions

Fractional-order advection–diffusion equations have matured from a mathematical curiosity into the standard language for anomalous transport. Their foundations in continuous-time random walks and Lévy motion are solid [1,2], their analytical toolbox is rich [18,68], and their numerical methods now span finite differences, spectral, finite element, meshless and lattice-Boltzmann families with proven convergence on smooth benchmarks [23,24,84,98]. Applications in hydrology, plasmas and heat transfer have demonstrated genuine predictive gains over classical models [1,6,119].



The discipline's central weakness is not a shortage of methods but a shortage of discipline: operator definitions proliferate faster than their physical content is established [27,29], and validation against real data remains the exception rather than the rule [28,110]. The field will progress fastest if the community standardizes derivations and benchmarks, if inverse problems and uncertainty quantification become routine parts of modelling studies [32,110], and if machine-learning methods are used to solve, rather than bypass, the underlying physics [31]. With those changes, fractional advection–diffusion models can fulfill their promise: parsimonious, physically transparent descriptions of the many real-world transport processes that refuse to be Gaussian.

References

1. Benson DA, Wheatcraft SW, Meerschaert MM. Application of a fractional advection-dispersion equation. *Water Resources Research*. 2000;36(6):1403-1412. doi:10.1029/2000wr900031
2. Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*. 2000;339(1):1-77. doi:10.1016/s0370-1573(00)00070-3
3. Benson DA, Schumer R, Meerschaert MM, Wheatcraft SW. Fractional dispersion, Lévy motion, and the MADE tracer tests. *Transport in Porous Media*. 2001;42(1-2):211-240. doi:10.1023/a:1006733002131
4. Baeumer B, Benson DA, Meerschaert MM, Wheatcraft SW. Subordinated advection-dispersion equation for contaminant transport. *Water Resources Research*. 2001;37(6):1543-1550. doi:10.1029/2000wr900409
5. Huang G, Huang Q, Zhan H. Evidence of one-dimensional scale-dependent fractional advection-dispersion. *Journal of Contaminant Hydrology*. 2006;85(1-2):53-71. doi:10.1016/j.jconhyd.2005.12.007
6. del-Castillo-Negrete D, Carreras BA, Lynch VE. Nondiffusive transport in plasma turbulence: a fractional diffusion approach. *Physical Review Letters*. 2005;94(6):065003. doi:10.1103/physrevlett.94.065003
7. Zimbardo G, Perri S. On the fractional diffusion-advection equation for fluids and plasmas. *Fluids*. 2019;4(2):62. doi:10.3390/fluids4020062
8. Zhang X, Lv M, Crawford JW, Young IM. The impact of boundary on the fractional advection-dispersion equation for solute transport in soil: defining the fractional dispersive flux with the Caputo derivatives. *Advances in Water Resources*. 2007;30(5):1205-1217. doi:10.1016/j.advwatres.2006.11.002
9. Lei D, Sun H, Zhang Y, Blaszczyk T, Yu Z. Upscaling solute transport in rough single-fractured media with matrix diffusion using a time fractional advection-dispersion equation. *Journal of Hydrology*. 2023;627:130280. doi:10.1016/j.jhydrol.2023.130280
10. Metzler R, Klafter J. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*. 2004;37(31):R161-R208. doi:10.1088/0305-4470/37/31/r01
11. Oldham KB, Spanier J. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press; 1974. doi:10.1016/s0076-5392(09)x6012-1
12. Podlubny I. Fractional derivatives and integrals. In: *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications (Mathematics in Science and Engineering)*. p. 41-119. 1999. doi:10.1016/s0076-5392(99)80021-6
13. Caputo M. Linear models of dissipation whose Q is almost frequency independent—II. *Geophysical Journal International*. 1967;13(5):529-539. doi:10.1111/j.1365-246x.1967.tb02303.x



14. Gorenflo R, Mainardi F. Continuous time random walk, Mittag-Leffler waiting time and fractional diffusion: mathematical aspects. In: *Anomalous Transport: Foundations and Applications*. p. 93-127. 2008. doi:10.1002/9783527622979.ch4
15. Gorenflo R, Mainardi F. Fractional diffusion processes: probability distributions and continuous time random walk. In: *Lecture Notes in Physics: Processes with Long-Range Correlations*. p. 148-166. 2003. doi:10.1007/3-540-44832-2_8
16. Benson DA, Wheatcraft SW, Meerschaert MM. The fractional-order governing equation of Lévy motion. *Water Resources Research*. 2000;36(6):1413-1423. doi:10.1029/2000wr900032
17. Schumer R, Benson DA, Meerschaert MM, Wheatcraft SW. Eulerian derivation of the fractional advection-dispersion equation. *Journal of Contaminant Hydrology*. 2001;48(1-2):69-88. doi:10.1016/s0169-7722(00)00170-4
18. Huang F, Liu F. The time fractional diffusion equation and the advection-dispersion equation. *The ANZIAM Journal*. 2005;46(3):317-330. doi:10.1017/s1446181100008282
19. Povstenko Y, Kyrylych T. Two approaches to obtaining the space-time fractional advection-diffusion equation. *Entropy*. 2017;19(7):297. doi:10.3390/e19070297
20. Guan W, Cao X. A numerical algorithm for the Caputo tempered fractional advection-diffusion equation. *Communications on Applied Mathematics and Computation*. 2021;3(1):41-59. doi:10.1007/s42967-020-00080-8
21. Zhang Y. Moments for tempered fractional advection-diffusion equations. *Journal of Statistical Physics*. 2010;139(5):915-939. doi:10.1007/s10955-010-9965-0
22. Zhang H, Shen S. The numerical simulation of space-time variable fractional order diffusion equation. *Numerical Mathematics: Theory, Methods and Applications*. 2013;6(4):571-585. doi:10.4208/nmtma.2013.y12107
23. Meerschaert MM, Tadjeran C. Finite difference approximations for fractional advection-dispersion flow equations. *Journal of Computational and Applied Mathematics*. 2004;172(1):65-77. doi:10.1016/j.cam.2004.01.033
24. Meerschaert MM, Tadjeran C. Finite difference approximations for two-sided space-fractional partial differential equations. *Applied Numerical Mathematics*. 2006;56(1):80-90. doi:10.1016/j.apnum.2005.02.008
25. Shen S, Liu F, Anh V. Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation. *Numerical Algorithms*. 2011;56(3):383-403. doi:10.1007/s11075-010-9393-x
26. Alobaidi AKL. Caputo-type fractional differential equations: recent advances and review. *International Journal of Research and Scientific Innovation*. 2026;13(7):77-94. doi:10.51244/ijrsi.2026.1307000006
27. Angstmann CN, Jacobs BA, Henry BI, Xu Z. Intrinsic discontinuities in solutions of evolution equations involving fractional Caputo-Fabrizio and Atangana-Baleanu operators. *Mathematics*. 2020;8(11):2023. doi:10.3390/math8112023
28. Stern R, Effenberger F, Fichtner H, Schäfer T. The space-fractional diffusion-advection equation: analytical solutions and critical assessment of numerical solutions. *Fractional Calculus and Applied Analysis*. 2014;17(1):171-190. doi:10.2478/s13540-014-0161-9



29. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*. 2014;264:65-70. doi:10.1016/j.cam.2014.01.002
30. Abdeljawad T. On conformable fractional calculus. *Journal of Computational and Applied Mathematics*. 2015;279:57-66. doi:10.1016/j.cam.2014.10.016
31. Momeni H, Yazdani Cherati A, Valinejad A. AFP-PINN: adaptive fractional polynomial-based physics-informed neural network for solving distributed-order time-fractional diffusion equations and its error analysis. *Physica Scripta*. 2025;100(7):076026. doi:10.1088/1402-4896/adeaf8
32. Slodička M. An inverse source problem in a variable-order time-fractional diffusion PDE. *Mathematics*. 2026;14(3):488. doi:10.3390/math14030488
33. Heymans N, Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheologica Acta*. 2006;45(5):765-771. doi:10.1007/s00397-005-0043-5
34. Atıcı FM, Chang S, Jonnalagadda JM. Grünwald-Letnikov fractional operators: from past to present. *Fractional Differential Calculus*. 2021;11(1):147-159. doi:10.7153/fdc-2021-11-10
35. Dorrego GA. An alternative definition for the k-Riemann-Liouville fractional derivative. *Applied Mathematical Sciences*. 2015;9:481-491. doi:10.12988/ams.2015.411893
36. Atangana A, Gómez-Aguilar JF. Numerical approximation of Riemann-Liouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu. *Numerical Methods for Partial Differential Equations*. 2018;34(5):1502-1523. doi:10.1002/num.22195
37. Dokuyucu MA, Dutta H. A fractional order model for Ebola virus with the new Caputo fractional derivative without singular kernel. *Chaos, Solitons & Fractals*. 2020;134:109717. doi:10.1016/j.chaos.2020.109717
38. Lavín-Delgado JE, Solís-Pérez JE, Gómez-Aguilar JF, Escobar-Jiménez RF. A new fractional-order mask for image edge detection based on Caputo-Fabrizio fractional-order derivative without singular kernel. *Circuits, Systems, and Signal Processing*. 2020;39(3):1419-1448. doi:10.1007/s00034-019-01200-3
39. Dokuyucu MA, Baleanu D, Çelik E. Analysis of Keller-Segel model with Atangana-Baleanu fractional derivative. *Filomat*. 2018;32(16):5633-5643. doi:10.2298/fil1816633d
40. Zhuang P, Liu F, Anh V, Turner I. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM Journal on Numerical Analysis*. 2009;47(3):1760-1781. doi:10.1137/080730597
41. Umarov S. Continuous time random walk models associated with distributed order diffusion equations. *Fractional Calculus and Applied Analysis*. 2015;18(3):821-837. doi:10.1515/fca-2015-0049
42. Rubbab Q, Mirza IA, Qureshi MZA. Analytical solutions to the fractional advection-diffusion equation with time-dependent pulses on the boundary. *AIP Advances*. 2016;6(7). doi:10.1063/1.4960108
43. Zoia A. Continuous-time random-walk approach to normal and anomalous reaction-diffusion processes. *Physical Review E*. 2008;77(4):041115. doi:10.1103/physreve.77.041115
44. Metzler R, Barkai E, Klafter J. Anomalous diffusion and relaxation close to thermal equilibrium: a fractional Fokker-Planck equation approach. *Physical Review Letters*. 1999;82(18):3563-3567. doi:10.1103/physrevlett.82.3563



45. Meerschaert MM, Mortensen J, Wheatcraft SW. Fractional vector calculus for fractional advection-dispersion. *Physica A: Statistical Mechanics and its Applications*. 2006;367:181-190. doi:10.1016/j.physa.2005.11.015
46. Meerschaert MM, Scheffler HP, Tadjeran C. Finite difference methods for two-dimensional fractional dispersion equation. *Journal of Computational Physics*. 2006;211(1):249-261. doi:10.1016/j.jcp.2005.05.017
47. Kelly JF, Meerschaert MM. The fractional advection-dispersion equation for contaminant transport. In: *Handbook of Fractional Calculus with Applications: Applications in Physics, Part B*. p. 129-150. 2019. doi:10.1515/9783110571721-006
48. Metzler R, Klafter J, Sokolov IM. Anomalous transport in external fields: continuous time random walks and fractional diffusion equations extended. *Physical Review E*. 1998;58(2):1621-1633. doi:10.1103/physreve.58.1621
49. Tsallis C, Lenzi EK. Anomalous diffusion: nonlinear fractional Fokker-Planck equation. *Chemical Physics*. 2002;284(1-2):341-347. doi:10.1016/s0301-0104(02)00557-8
50. Schumer R, Benson DA, Meerschaert MM, Baeumer B. Fractal mobile/immobile solute transport. *Water Resources Research*. 2003;39(10). doi:10.1029/2003wr002141
51. Schumer R, Benson DA, Meerschaert MM, Baeumer B. Multiscaling fractional advection-dispersion equations and their solutions. *Water Resources Research*. 2003;39(1). doi:10.1029/2001wr001229
52. Povstenko Y. Generalized boundary conditions for the time-fractional advection diffusion equation. *Entropy*. 2015;17(6):4028-4039. doi:10.3390/e17064028
53. Alkandari M, Loutchko D, Luchko Y. A tempered fractional diffusion equation with Sonin kernel as anomalous diffusion model. *Communications in Nonlinear Science and Numerical Simulation*. 2026;162:110262. doi:10.1016/j.cnsns.2026.110262
54. McLean W, Mustapha K, Ali R, Knio O. Well-posedness of time-fractional advection-diffusion-reaction equations. *Fractional Calculus and Applied Analysis*. 2019;22(4):918-944. doi:10.1515/fca-2019-0050
55. Kemppainen J. Existence and uniqueness of the solution for a time-fractional diffusion equation. *Fractional Calculus and Applied Analysis*. 2011;14(3):411-417. doi:10.2478/s13540-011-0025-5
56. Silvestre L. Hölder estimates for advection fractional-diffusion equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*. 2012;843-855. doi:10.2422/2036-2145.201009_004
57. Dimache AN, Groza G, Jianu M, Iancu I. Existence and uniqueness of solution represented as fractional power series for the fractional advection-dispersion equation. *Symmetry*. 2024;16(9):1137. doi:10.3390/sym16091137
58. Povstenko Y, Klekot J. The Dirichlet problem for the time-fractional advection-diffusion equation in a line segment. *Boundary Value Problems*. 2016. doi:10.1186/s13661-016-0597-4
59. Mahmoud EI, Aleroev TS. Boundary value problem of space-time fractional advection diffusion equation. *Mathematics*. 2022;10(17):3160. doi:10.3390/math10173160
60. Malysheva T, White LW. An advection-diffusion equation with a generalized advection term: well-posedness analysis and examples. *Examples and Counterexamples*. 2024;6:100159. doi:10.1016/j.exco.2024.100159



61. Zhou Y. Well-posedness of fractional diffusion equations. In: Fractional Diffusion and Wave Equations. p. 21-80. 2024. doi:10.1007/978-3-031-74031-2_2
62. Tang Q. On an optimal control problem of time-fractional advection-diffusion equation. Discrete and Continuous Dynamical Systems - B. 2020;25(2):761-779. doi:10.3934/dcdsb.2019266
63. Gill V, Singh J, Singh Y. Analytical solution of generalized space-time fractional advection-dispersion equation via coupling of Sumudu and Fourier transforms. Frontiers in Physics. 2019;6:151. doi:10.3389/fphy.2018.00151
64. Huang W. Analytical time-dependent solutions of modelling of metal electrodeposits. Europhysics Letters. 2026;154(4):42002. doi:10.1209/0295-5075/ae647a
65. Kundu S. Two-parameter Mittag-Leffler solution of space fractional advection-diffusion equation for sediment suspension in turbulent flows. Journal of Environmental Engineering. 2018;144(8). doi:10.1061/(asce)ee.1943-7870.0001416
66. Jadhav CP, Dale T, Chinchane V. On Dirichlet problem of time-fractional advection-diffusion equation. Journal of Fractional Calculus and Nonlinear Systems. 2023;4(2):1-13. doi:10.48185/jfcns.v4i2.861
67. Bologna M, Tsallis C, Grigolini P. Anomalous diffusion associated with nonlinear fractional derivative Fokker-Planck-like equation: exact time-dependent solutions. Physical Review E. 2000;62(2):2213-2218. doi:10.1103/physreve.62.2213
68. Yildirim A, Koçak H. Homotopy perturbation method for solving the space-time fractional advection-dispersion equation. Advances in Water Resources. 2009;32(12):1711-1716. doi:10.1016/j.advwatres.2009.09.003
69. Singh MK, Chatterjee A. Solution of one-dimensional space- and time-fractional advection-dispersion equation by homotopy perturbation method. Acta Geophysica. 2017;65(2):353-361. doi:10.1007/s11600-017-0035-8
70. Singh MK, Chatterjee A, Singh VP. Solution of one-dimensional time fractional advection dispersion equation by homotopy analysis method. Journal of Engineering Mechanics. 2017;143(9). doi:10.1061/(asce)em.1943-7889.0001318
71. Kumar S, Pandey RK, Sharma S, Singh H. Legendre collocation method for generalized fractional advection diffusion equation. In: Special Functions in Fractional Calculus and Engineering. p. 111-125. 2023. doi:10.1201/9781003368069-5
72. Kumar S, Zeidan D. An efficient Mittag-Leffler kernel approach for time-fractional advection-reaction-diffusion equation. Applied Numerical Mathematics. 2021;170:190-207. doi:10.1016/j.apnum.2021.07.025
73. Kumar S, Ahmad B. A new numerical study of space-time fractional advection-reaction-diffusion equation with Rabotnov fractional-exponential kernel. Numerical Methods for Partial Differential Equations. 2021. doi:10.1002/num.22647
74. Sene N. Double integration for the fractional advection-dispersion equation with nonsingular derivative. In: Applications of Fractional Calculus to Modeling in Dynamics and Chaos. p. 331-352. 2022. doi:10.1201/9781003006244-13
75. Li C, Deng W. A new family of difference schemes for space fractional advection diffusion equation. Advances in Applied Mathematics and Mechanics. 2017;9(2):282-306. doi:10.4208/aamm.2015.m1069



76. Abdel-Rehim EA. Implicit difference scheme of the space-time fractional advection diffusion equation. *Fractional Calculus and Applied Analysis*. 2015;18(6):1452-1469. doi:10.1515/fca-2015-0084
77. Momani S, Abu Rqayiq A, Baleanu D. A nonstandard finite difference scheme for two-sided space-fractional partial differential equations. *International Journal of Bifurcation and Chaos*. 2012;22(4):1250079. doi:10.1142/s0218127412500794
78. Salama FM. A parallel Krylov subspace iterative scheme for variable-order fractional advection-diffusion-reaction equation. *Fractal and Fractional*. 2026;10(6):378. doi:10.3390/fractalfract10060378
79. Salama FM. An efficient numerical treatment of variable-order time fractional advection-diffusion equation. *Computational and Applied Mathematics*. 2026;45(1). doi:10.1007/s40314-025-03418-1
80. Ahmadinia M, Abbasi M, Hadi P. An intelligent non-uniform mesh to improve errors of a stable numerical method for time-tempered fractional advection-diffusion equation with weakly singular solution. *The Journal of Supercomputing*. 2024;80(18):26280-26307. doi:10.1007/s11227-024-06442-w
81. Ali U, Sohail M, Abdullah FA. An efficient numerical scheme for variable-order fractional sub-diffusion equation. *Symmetry*. 2020;12(9):1437. doi:10.3390/sym12091437
82. Brahim N. Numerical approach of the nonlinear reaction-advection-diffusion equation with time-space conformable fractional derivatives. *AIP Conference Proceedings*. 2021;2334:060012. doi:10.1063/5.0042459
83. Chen W, Lü S, Chen H, Liu H. Optimal error estimate of the Legendre spectral approximation for space-fractional reaction-advection-diffusion equation. *Advances in Difference Equations*. 2018. doi:10.1186/s13662-018-1572-4
84. Zaky MA. Finite difference/fractional Petrov-Galerkin spectral method for linear time-space fractional reaction-diffusion equation. *Mathematics*. 2025;13(11):1864. doi:10.3390/math13111864
85. Ma H, Yang Y. Jacobi spectral collocation method for the time variable-order fractional mobile-immobile advection-dispersion solute transport model. *East Asian Journal on Applied Mathematics*. 2016;6(3):337-352. doi:10.4208/eajam.141115.060616a
86. Gupta R, Kumar S. Chebyshev spectral method for the variable-order fractional mobile-immobile advection-dispersion equation arising from solute transport in heterogeneous media. *Journal of Engineering Mathematics*. 2023;142(1). doi:10.1007/s10665-023-10288-1
87. Marasi H, Derakhshan M. A composite collocation method based on the fractional Chelyshkov wavelets for distributed-order fractional mobile-immobile advection-dispersion equation. *Mathematical Modelling and Analysis*. 2022;27(4):590-609. doi:10.3846/mma.2022.15311
88. Vo TN, Razzaghi M, Toan PT. A numerical method for solving variable-order fractional diffusion equations using fractional-order Taylor wavelets. *Numerical Methods for Partial Differential Equations*. 2021;37(3):2668-2686. doi:10.1002/num.22761
89. Azin H, Mohammadi F, Heydari MH. A hybrid method for solving time fractional advection-diffusion equation on unbounded space domain. *Advances in Difference Equations*. 2020;2020(1). doi:10.1186/s13662-020-03053-6
90. Hussain M, Haq S. A hybrid radial basis functions collocation technique to numerically solve fractional advection-diffusion models. *Numerical Methods for Partial Differential Equations*. 2020;36(6):1254-1279. doi:10.1002/num.22472



91. Mohebbi A, Abbaszadeh M, Dehghan M. Solution of two-dimensional modified anomalous fractional sub-diffusion equation via radial basis functions (RBF) meshless method. *Engineering Analysis with Boundary Elements*. 2014;38:72-82. doi:10.1016/j.enganabound.2013.09.015
92. Mirzaee F, Samadyar N. Combination of finite difference method and meshless method based on radial basis functions to solve fractional stochastic advection-diffusion equations. *Engineering with Computers*. 2020;36(4):1673-1686. doi:10.1007/s00366-019-00789-y
93. Abbaszadeh M, Dehghan M. Meshless upwind local radial basis function-finite difference technique to simulate the time-fractional distributed-order advection-diffusion equation. *Engineering with Computers*. 2021;37(2):873-889. doi:10.1007/s00366-019-00861-7
94. Kamran, Khan S, Alhazmi SE, Alotaibi FM, Ferrara M, Ahmadian A. On the numerical approximation of mobile-immobile advection-dispersion model of fractional order arising from solute transport in porous media. *Fractal and Fractional*. 2022;6(8):445. doi:10.3390/fractalfract6080445
95. Tajadodi H. Variable-order Mittag-Leffler fractional operator and application to mobile-immobile advection-dispersion model. *Alexandria Engineering Journal*. 2022;61(5):3719-3728. doi:10.1016/j.aej.2021.09.007
96. Pandey P, Gómez-Aguilar JF. On solution of a class of nonlinear variable order fractional reaction-diffusion equation with Mittag-Leffler kernel. *Numerical Methods for Partial Differential Equations*. 2021;37(2):998-1011. doi:10.1002/num.22563
97. Zheng Y, Li C, Zhao Z. A note on the finite element method for the space-fractional advection diffusion equation. *Computers & Mathematics with Applications*. 2010;59(5):1718-1726. doi:10.1016/j.camwa.2009.08.071
98. Dehghan M, Abbaszadeh M. A finite difference/finite element technique with error estimate for space fractional tempered diffusion-wave equation. *Computers & Mathematics with Applications*. 2018;75(8):2903-2914. doi:10.1016/j.camwa.2018.01.020
99. Bin Jebreen H, Wang H. On the effective method for the space-fractional advection-diffusion equation by the Galerkin method. *AIMS Mathematics*. 2024;9(9):24143-24162. doi:10.3934/math.20241173
100. Qasmi MHA, Bukhari SAM, Alshabanat A, Ali U. Modeling and numerical investigation of space-fractional advection diffusion models by Galerkin's weighted residual method. *Journal of Mathematics*. 2026;2026(1):9205345. doi:10.1155/jom/9205345
101. Dwivedi KD, Rajeev, Das S, Baleanu D. Numerical solution of nonlinear space-time fractional-order advection-reaction-diffusion equation. *Journal of Computational and Nonlinear Dynamics*. 2020;15(6). doi:10.1115/1.4046879
102. Ta TTM. A priori error analysis of a space-time finite element method for advection-diffusion equations. *Acta Mathematica Vietnamica*. 2026;51(1):85-102. doi:10.1007/s40306-025-00596-2
103. Cartalade A, Younsi A, Néel MC. Multiple-relaxation-time lattice Boltzmann scheme for fractional advection-diffusion equation. *Computer Physics Communications*. 2019;234:40-54. doi:10.1016/j.cpc.2018.08.005
104. Ren J, Song J. Lattice Boltzmann method for tempered space-fractional advection-diffusion equation with applications to tempered fractional transport. *Physics of Fluids*. 2025;37(8). doi:10.1063/5.0278921



105. Allwright A, Atangana A. Upwind-based numerical approximation of a space-time fractional advection-dispersion equation for groundwater transport within fractured systems. In: *Studies in Systems, Decision and Control*. p. 309-341. 2019. doi:10.1007/978-3-030-11662-0_18
106. Golbabai A, Nikan O, Nikazad T. Numerical investigation of the time fractional mobile-immobile advection-dispersion model arising from solute transport in porous media. *International Journal of Applied and Computational Mathematics*. 2019;5(3). doi:10.1007/s40819-019-0635-x
107. Rundell W, Zhang Z. Fractional diffusion: recovering the distributed fractional derivative from overposed data. *Inverse Problems*. 2017;33(3):035008. doi:10.1088/1361-6420/aa573e
108. Mehdinejadiani B. Estimating the solute transport parameters of the spatial fractional advection-dispersion equation using Bees Algorithm. *Journal of Contaminant Hydrology*. 2017;203:51-61. doi:10.1016/j.jconhyd.2017.06.004
109. Wei H, Dang W, Wei S. Parameter identification of solute transport with spatial fractional advection-dispersion equation via Tikhonov regularization. *Optik*. 2017;129:8-14. doi:10.1016/j.ijleo.2016.10.052
110. Janno J, Kinash N. Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements. *Inverse Problems*. 2018;34(2):025007. doi:10.1088/1361-6420/aaa0f0
111. Sun C, Liu J. An inverse source problem for distributed order time-fractional diffusion equation. *Inverse Problems*. 2020;36(5):055008. doi:10.1088/1361-6420/ab762c
112. Artyushin AN. An inverse problem of recovering the variable order of the derivative in a fractional diffusion equation. *Siberian Mathematical Journal*. 2023;64(4):796-806. doi:10.1134/s003744662304002x
113. Tatar S, Ulusoy S. An inverse problem for a nonlinear diffusion equation with time-fractional derivative. *Journal of Inverse and Ill-posed Problems*. 2017;25(2):185-193. doi:10.1515/jiip-2015-0100
114. Singh A, Das S. Study and analysis of spatial-time nonlinear fractional-order reaction-advection-diffusion equation. *Journal of Porous Media*. 2019;22(7):787-798. doi:10.1615/jpormedia.2019025907
115. Moradi G, Mehdinejadiani B. Modelling solute transport in homogeneous and heterogeneous porous media using spatial fractional advection-dispersion equation. *Soil and Water Research*. 2018;13(1):18-28. doi:10.17221/245/2016-swr
116. del-Castillo-Negrete D, Carreras BA, Lynch VE. Fractional diffusion in plasma turbulence. *Physics of Plasmas*. 2004;11(8):3854-3864. doi:10.1063/1.1767097
117. Effenberger F, Aerdker S, Merten L, Fichtner H. Superdiffusion of energetic particles at shocks: a fractional diffusion and Lévy flight model of spatial transport. *Astronomy & Astrophysics*. 2024;686:A219. doi:10.1051/0004-6361/202449334
118. Zimbardo G, Malara F, Perri S. Energetic particle superdiffusion in solar system plasmas: which fractional transport equation?. *Symmetry*. 2021;13(12):2368. doi:10.3390/sym13122368
119. Xu G, Wang J. Analytical solution of time fractional Cattaneo heat equation for finite slab under pulse heat flux. *Applied Mathematics and Mechanics (English Edition)*. 2018;39(10):1465-1476. doi:10.1007/s10483-018-2375-8
120. Zhao J, Zheng L, Chen X, Zhang X, Liu F. Unsteady Marangoni convection heat transfer of fractional Maxwell fluid with Cattaneo heat flux. *Applied Mathematical Modelling*. 2017;44:497-507. doi:10.1016/j.apm.2017.02.021



121. Shen M, Chen L, Zhang M, Liu F. A renovated Buongiorno's model for unsteady Sisko nanofluid with fractional Cattaneo heat flux. *International Journal of Heat and Mass Transfer*. 2018;126:277-286. doi:10.1016/j.ijheatmasstransfer.2018.05.131
122. Li SN, Cao BY. Anomalous heat diffusion from fractional Fokker-Planck equation. *Applied Mathematics Letters*. 2020;99:105992. doi:10.1016/j.aml.2019.07.023
123. Pawar A, Pardasani KR. Fractional-order reaction-diffusion model to study the dysregulatory impacts of superdiffusion and memory on neuronal calcium and IP3 dynamics. *The European Physical Journal Plus*. 2023;138(9). doi:10.1140/epjp/s13360-023-04410-6
124. Tour G, Thakoor N, Tangman DY. Spectrally accurate option pricing under the time fractional Black-Scholes model. *The ANZIAM Journal*. 2021;63:228-248. doi:10.21914/anziamj.v63.15412
125. Pariyar S, Poudel EP. Prediction of air pollution levels through time-fractional advection-diffusion equation. *Baneshwor Campus Journal of Academia*. 2025;4:177-187. doi:10.3126/bcja.v4i1.90172