



Structural Invalidity of the Modified Chain Rule in He et al.

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Abstract

This work re-examines the modified chain-rule identity proposed by He, Elagan, and Li (2012). Although they correctly demonstrate that the previously used chain rule is invalid, the replacement formula containing a geometric scaling factor does not resolve the underlying analytic inconsistency. Both identities share the same local-product structure, which is incompatible with the integral definition of Jumarie's modified Riemann–Liouville derivative. Multiple counterexamples show that no fixed geometric constant can make the modified identity valid. Consequently, the reduction procedure in Section 5 of the original paper lacks mathematical justification, and subsequent works relying on this chain rule should be revisited. This clarification helps prevent further propagation of incorrect chain-rule assumptions in fractional-calculus research.

Keywords: Chain rule, Jumarie, Riemann–Liouville derivative, fractional calculus, fractional complex transform.

1. Introduction

The chain rule is one of the most fundamental identities in classical calculus. It allows derivatives of composite functions to be expressed in terms of derivatives of their components, enabling change of variables in

differential equations, reduction of nonlinear problems to simpler forms, construction of exact solutions via transforms, and consistency between different coordinate systems. In ordinary calculus, the chain rule guarantees that the derivative behaves predictably under composition, and many solution techniques for differential equations rely critically on this property.

In fractional calculus [1-2], and in particular for Jumarie's modified Riemann–Liouville derivative [3], the need for a valid chain rule is even more critical. Methods such as the fractional complex transform aim to convert a fractional differential equation into an ordinary differential equation by introducing a new variable. This reduction step requires a valid chain rule for the fractional derivative being used; without such a rule, the transform loses its mathematical foundation and any solutions obtained through it may not genuinely solve the original fractional equation.

He et al. [4–12] examine the fractional complex transform and the chain rule for Jumarie's modified Riemann–Liouville derivative. In Section 4 of [12], they explicitly state, after their Eq. (12), that “This chain rule is invalid.” They support this conclusion with a counterexample involving $u(t) = t^2$ and $s(x) = x^\beta$ (their Eqs. (13)–(16)). The invalidated rule in [12] is

$$D_t^\alpha u(s(t)) = u'(s(t)) D_t^\alpha s(t). \quad (12)$$

However, they proposed replacement for Eq. (12)

$$D_t^\alpha u(s(t)) = o' u'(s(t)) D_t^\alpha s(t), \quad (22)$$



It is derived from geometric considerations (their Eqs. (18)–(21)), where $o' = k/k_x$ is a constant determined by the projection of a fractal curve. The purpose of this Comment is to show, in a rigorous way, that the inclusion of o' does not alter the analytic structure of the chain rule and therefore does not resolve the inconsistency demonstrated in [12].

Moreover, fractional derivatives are nonlocal operators whose values at a point depend on the entire past history of the function. A composition $u(s(t))$ intertwines the histories of both u and s , so any legitimate chain rule must respect this nonlocal structure. A naive pointwise product formula, such as (12) or its modified version (22), ignores this dependence and risks producing incorrect transformations and solutions. Therefore, clarifying the validity of chain-rule-type identities in fractional calculus is essential not only for the specific results of He et al. but also for the reliability of a broad class of analytical methods that build on these identities.

2. Structural Invalidity of the Modified Chain Rule

Jumarie's modified Riemann–Liouville derivative of order $0 < \alpha < 1$ is defined by:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau. \quad (2.1)$$

This operator is defined via an integral over $[0, t]$, so its value at t depends on the entire history of $f(\tau)$ for $0 \leq \tau \leq t$.

Lemma 2.1 (Nonlocality vs. Local Product Rules)

Let u and s be differentiable functions. No identity of the form

$$D_t^\alpha (u \circ s)(t) = C u'(s(t)) D_t^\alpha s(t), \quad (2.2)$$

can hold for all u, s and all $0 < \alpha < 1$, where C is any fixed constant independent of u, s, α .

Proof. The right-hand side of (2.2) depends only on the values of $u'(s(t))$ and $D_t^\alpha s(t)$ at the single point t . The left-hand side, by (2.1), depends on the integral of $u(s(\tau))$ over $0 \leq \tau \leq t$. If (2.2) were valid for all u, s , the integral operator (2.1) would reduce to a pointwise operator on all compositions $u \circ s$, contradicting its definition. Hence no constant C can make (2.2) universally valid. ■

By Lemma 2.1, any identity of the form (22) is structurally invalid under (2.1), regardless of the value of o' . The geometric factor does not change the fact that the right-hand side is a local product, incompatible with the integral operator on the left. Thus the defect in (12) is structural and cannot be repaired by introducing a constant multiplier.

3. Counterexamples

To demonstrate the failure of the modified chain rule, we examine several representative compositions, including polynomial, exponential, and logarithmic functions. In each case, equating the true fractional derivative of $u(s(x))$ with the proposed identity requires o' to depend on parameters such as α, β , or x . Because He et al. [12] define o' as a fixed geometric constant, the identity cannot hold. These counterexamples confirm that the defect is structural and cannot be repaired by scaling.

Example 1: Quadratic Composition (He et al.)

Let

$$u(t) = t^2, \quad s(x) = x^\beta, \quad \beta > 0. \quad (3.1)$$

For Jumarie's derivative of a power function,

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad (3.2)$$



we obtain

$$D_x^\alpha(u(s(x))) = D_x^\alpha x^{2\beta} = \frac{\Gamma(2\beta+1)}{\Gamma(2\beta+1-\alpha)} x^{2\beta-\alpha}. \quad (3.3)$$

On the other hand,

$$u'(s(x)) = 2s(x) = 2x^\beta, \quad (3.4)$$

and

$$D_x^\alpha s(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.5)$$

Thus the right-hand side of (22) becomes

$$o' u'(s(x)) D_x^\alpha s(x) = o' 2 \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{2\beta-\alpha}. \quad (3.6)$$

Equating (3.3) and (3.6) requires

$$o' = \frac{\Gamma(2\beta+1) \Gamma(\beta+1-\alpha)}{2 \Gamma(\beta+1) \Gamma(2\beta+1-\alpha)}. \quad (3.7)$$

The right-hand side depends on α and β . In contrast, o' is defined in [12] as a fixed geometric constant determined solely by the projection ratio k/k_x , independent of α, β . Hence no fixed constant can satisfy (3.7).

Example 2: Exponential Composition

Let

$$u(t) = e^t, \quad s(x) = x^\beta. \quad (3.8)$$

Then $u(s(x)) = e^{x^\beta}$. Expanding e^{x^β} as a power series and using (3.2),

$$D_x^\alpha e^{x^\beta} = \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma(n\beta+1-\alpha)} x^{n\beta-\alpha}. \quad (3.9)$$

On the other hand,

$$u'(s(x)) = e^{x^\beta}, \quad D_x^\alpha s(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.10)$$

Thus

$$o' u'(s(x)) D_x^\alpha s(x) = o' e^{x^\beta} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.11)$$

Equating (3.9) and (3.11) gives

$$o' = \frac{\sum_{n=0}^{\infty} \frac{\beta^n x^{n\beta}}{\Gamma(n\beta+1-\alpha)}}{e^{x^\beta} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}}. \quad (3.12)$$

This depends on x, α, β , so no fixed geometric constant o' can satisfy (3.12).

Example 3: Logarithmic Composition

Let

$$u(t) = \ln(1+t), \quad s(x) = x^\beta. \quad (3.13)$$

Then $u(s(x)) = \ln(1+x^\beta)$. Using known fractional-derivative formulas,



$$D_x^\alpha \ln(1+x^\beta) = \frac{x^{\beta-\alpha}}{\Gamma(1-\alpha)} {}_2F_1(1, 1-\alpha; 2-\alpha; -x^\beta). \quad (3.14)$$

On the other hand,

$$u'(s(x)) = \frac{1}{1+x^\beta}, \quad D_x^\alpha s(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.15)$$

Thus

$$o' u'(s(x)) D_x^\alpha s(x) = o' \frac{1}{1+x^\beta} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.16)$$

Equating (3.14) and (3.16) requires

$$o' = \frac{{}_2F_1(1, 1-\alpha; 2-\alpha; -x^\beta)}{\Gamma(1-\alpha) \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \frac{1}{1+x^\beta}}. \quad (3.17)$$

Again o' depends on x, α, β , contradicting the assumption that o' is a fixed geometric constant.

Example 4: Cubic Polynomial Composition

Let

$$u(t) = t^3, \quad s(x) = x^\beta. \quad (3.18)$$

Then $u(s(x)) = x^{3\beta}$. Using (3.2),

$$D_x^\alpha x^{3\beta} = \frac{\Gamma(3\beta+1)}{\Gamma(3\beta+1-\alpha)} x^{3\beta-\alpha}. \quad (3.19)$$

On the other hand,

$$u'(s(x)) = 3x^{2\beta}, \quad D_x^\alpha s(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}. \quad (3.20)$$

Thus

$$o' u'(s(x)) D_x^\alpha s(x) = o' 3 \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{3\beta-\alpha}. \quad (3.21)$$

Equating (3.19) and (3.21) requires

$$o' = \frac{\Gamma(3\beta+1) \Gamma(\beta+1-\alpha)}{3 \Gamma(\beta+1) \Gamma(3\beta+1-\alpha)}. \quad (3.22)$$

This depends on α, β , so again no fixed geometric constant o' can satisfy (3.22).

Taken together, Examples 1–4 show that for a wide class of functions—polynomials of different degrees, exponentials, and logarithms—the identity (22) can only hold if o' depends on α, β and, in some cases, on x . Since He et al. define o' as a fixed geometric constant determined solely by the projection ratio k/k_x , Eq. (22) cannot be valid under Jumarie's definition. It shows that no fixed geometric constant can repair the chain rule.

4. Implications for the Fractional Complex Transform

We now examine the consequences of the failure of (22) for the reduction procedure used in Section 5 of [12]. Under Jumarie's definition (2.1), the operator D_t^α is integral in nature:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau. \quad (4.1)$$

Thus, for any differentiable f ,



$$D_t^\alpha f(t) \text{ depends on } f(\tau) \text{ for all } 0 \leq \tau \leq t. \quad (4.2)$$

Lemma 4.1 (No Constant Can Repair the Chain Rule)

Let u and s be differentiable. No constant o' independent of u, s, α can make

$$D_t^\alpha (u \circ s)(t) = o' u'(s(t)) D_t^\alpha s(t) \quad (4.3)$$

valid for all choices of u, s and $0 < \alpha < 1$.

Proof. The right-hand side of (4.3) depends only on $u'(s(t))$ and $D_t^\alpha s(t)$ at the point t . The left-hand side, by (4.1), depends on the integral of $u(s(\tau))$ over $[0, t]$. If (4.3) were valid for all u, s , the integral operator (4.1) would reduce to a pointwise operator on all compositions $u \circ s$, contradicting its definition. Hence no constant o' can make (4.3) universally valid.

■

In Section 5 of [12], Eq. (22) is invoked to reduce the fractional relaxation equation

$$D_t^\alpha u(t) + Bu(t) = 0$$

to an ordinary differential equation via a change of variable. The validity of this reduction requires (22) to hold for the specific functions used. By Lemma 4.2 and the explicit counterexamples in Section 3, (22) cannot be valid for any fixed geometric constant o' . Therefore the reduction step in Section 5 is not mathematically justified.

The final closed-form solution (their Eq. (36)) for the fractional relaxation equation is consistent with results obtained by other methods. The result itself may be correct, but its derivation through Eq. (22) is not rigorous, and a self-contained proof by an alternative route would be required to validate it independently of the incorrect chain rule.

The fractional complex transform relies on a chain-rule identity to reduce fractional differential equations to ordinary differential equations. Because the modified chain rule is invalid, the reduction used in Section 5 of He et al. is not mathematically justified. Although the final solution obtained in their paper may coincide with known results, the derivation lacks rigor and requires independent verification. Any subsequent works relying on the same chain-rule identity inherit this flaw and should be re-evaluated.

5. Discussion

The analysis above clarifies the mathematical status of the chain-rule-type identities used in He et al. (2012). Under Jumarie's definition (2.1), the fractional derivative $D_t^\alpha f$ is determined by an integral over $[0, t]$, and cannot be represented by a pointwise product involving only $u'(s(t))$ and $D_t^\alpha s(t)$. The original identity (12) fails for this structural reason. The modified identity (22) retains the same form and, as Lemma 4.2 and the multiple counterexamples in Section 3 show, cannot be repaired by introducing a constant multiplier.

A deeper issue emerges when considering the role of the chain rule in fractional calculus. In classical calculus, the chain rule is not merely a computational convenience; it is a structural identity that ensures compatibility between differentiation and composition. Many analytical techniques—such as variable transformations, similarity reductions, and nonlinear mappings—depend critically on the chain rule. When the derivative operator is local, as in the classical case, the chain rule naturally reflects the local dependence of the derivative on the function's behavior at a single point.

However, fractional derivatives are inherently nonlocal. Jumarie's modified Riemann–Liouville derivative explicitly incorporates the entire history of the function through an integral over $[0, t]$. This nonlocality means that the derivative of a composite function $u(s(t))$ depends on the full past behavior of both u and s , not merely on their instantaneous values at t . Any valid chain rule for such an operator must therefore account for this historical dependence. A pointwise product formula—whether in the form of (12) or the modified form (22)—cannot capture this nonlocal structure.



The geometric argument introduced in He et al. attempts to justify the modified chain rule by appealing to fractal projections and scaling factors. While geometric intuition can be valuable in fractional calculus, the specific construction in [12] does not alter the analytic nature of the derivative operator. The constant o' , defined as a projection ratio k/k_x , is independent of the functional parameters α , β , and x . However, as demonstrated in Section 3, any constant that would make (22) valid must depend on these parameters. This mismatch shows that the geometric reasoning in [12] cannot resolve the analytic inconsistency.

The implications for the fractional complex transform are significant. This transform is intended to convert fractional differential equations into ordinary differential equations by introducing a new variable $s(t)$. Such a reduction requires a valid chain rule for the fractional derivative. If the chain rule is invalid, the transform does not preserve the structure of the original equation, and the resulting ordinary differential equation may not correspond to the fractional equation being studied. In Section 5 of [12], they rely on (22) to reduce the fractional relaxation equation to an ordinary differential equation. Because (22) is invalid, this reduction step lacks mathematical justification.

It is important to emphasize that the invalidity of the chain rule does not necessarily invalidate the final solution obtained in [12]. As noted earlier, the closed-form solution for the fractional relaxation equation agrees with results derived by other methods. The issue lies not with the solution itself, but with the logical path used to obtain it. A correct solution derived from an incorrect method still requires independent verification. In this case, a self-contained derivation that does not rely on the flawed chain rule would be necessary to establish the result rigorously.

The broader consequence is that any subsequent work relying on the modified chain rule (22) inherits the same structural flaw. Because the chain rule is often used in fractional calculus to justify variable transformations, similarity reductions, and analytic solution methods, an incorrect chain rule can propagate errors across multiple papers. Researchers using Jumarie's derivative must therefore exercise caution when applying chain-rule-type identities and ensure that their methods respect the nonlocal nature of the operator.

In summary, the discussion highlights a fundamental tension between geometric intuition and analytic rigor in fractional calculus. While geometric interpretations can offer valuable insights, they cannot override the integral structure of fractional derivatives. Any chain rule compatible with Jumarie's definition must reflect this nonlocality. The modified identity (22) does not, and therefore cannot serve as a valid analytic tool for transforming fractional differential equations.

6. Conclusion

The modified chain-rule identity proposed by He et al. (2012) is structurally invalid under Jumarie's definition of the fractional derivative. No fixed geometric constant can make the identity hold for all functions, and the reduction procedure relying on this identity is not rigorous. This clarification is essential for preventing further propagation of incorrect methodology in fractional-calculus research. Future work should focus on developing reduction methods compatible with the nonlocal structure of fractional operators.

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