



The Camassa–Holm Equation: A Critical Review of Integrable Structure, Analysis, Peakons, and Applications to 2026

Dr. Weiguang Huang

DrHuang.com, Sydney, NSW 2033, Australia

ORCID: 0000-0002-5766-4516

How to Cite this Article:

Huang, W. (2026). The Camassa–Holm Equation: A Critical Review of Integrable Structure, Analysis, Peakons, and Applications to 2026. International Journal of Creative and Open Research in Engineering and Management, 2(10), 1-9.
<https://doi.org/10.55041/ijcope.v2i9.253>

License:

This article is published under the terms of the Creative Commons Attribution 4.0 International License (CC BY 4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

© The Author(s). Published by International Journal of Creative and Open Research in Engineering and Management.



<https://doi.org/10.55041/ijcope.v2i9.253>

Abstract

The Camassa–Holm (CH) equation, introduced in 1993 as a model of shallow-water wave propagation, has become one of the most influential integrable partial differential equations of the past three decades. It is celebrated for three properties that rarely coexist: it is completely integrable, it is a physically relevant approximation of the water-wave problem, and it admits nonsmooth solitary waves, the celebrated peakons, that interact elastically and are orbitally stable. This review provides a critical, evidence-based synthesis of the field through 2026. We first trace the equation's physical derivation and its hydrodynamic relevance. We then examine its integrable structure — Lax pair, bi-Hamiltonian formulation, inverse scattering, and the string-equation connection — and the rich families of solutions from peakons and multipeakons to cuspons, solitary waves, and periodic waves. The analytical theory of well-posedness, wave breaking, global existence, and uniqueness of conservative and dissipative weak solutions is appraised in detail, as are recent results on ill-posedness in critical spaces, stochastic forcing, and boundary control. We review the numerical literature, including convergent finite-difference and spectral schemes, adaptive upwinding, and the growing fractional-order and data-driven literature. Related integrable systems — the Degasperis–Procesi equation, the Novikov equation, the modified Camassa–Holm equation, and their two-component and multi-component generalizations — are compared critically. Finally, we identify persistent open problems: sharp uniqueness of weak solutions, the role of the peakon in the global regularity theory, rigorous control of wave breaking under noise, and the physical status of fractional and regularized variants. A concrete research agenda closes the review.

Keywords: Camassa–Holm equation; peakons; integrable systems; inverse scattering; wave breaking; well-posedness; Degasperis–Procesi equation; Novikov equation; fractional Camassa–Holm equation.

1. Introduction

The Camassa–Holm equation describes the unidirectional propagation of shallow-water waves over a flat bottom. For a real-valued function $u(x, t)$ it is usually written in the form

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

or, with $m = u - u_{xx}$,

$$m_t + um_x + 2u_x m = 0,$$

where the parameter $\kappa \geq 0$ encodes linear dispersion [1]. In the case $\kappa = 0$ the equation is scale invariant, and in the case $\kappa > 0$ it is the form relevant for water waves of small amplitude [2]. The equation was derived by Camassa and Holm from the Green–Naghdi shallow-water equations by an asymptotic expansion that retains nonlinear dispersion [1]. Its most striking prediction was the existence of peaked solitons — traveling waves $u = c e^{-|x-ct|}$ with a discontinuous derivative at the crest — that interact elastically like classical solitons but are not smooth [1,3]. Camassa and Holm named these waves peakons and showed numerically that they collide elastically [1], a property later put on



a rigorous footing for multipeakon dynamics by Alber, Camassa, Holm, and Marsden through the billiard correspondence [3].

The CH equation emerged from a wider program on integrable shallow-water models. Fuchssteiner and Fokas had earlier observed that the equation belongs to a family obtainable from the symmetry toolbox of nonlinear evolution equations [4], and Fuchssteiner traced the underlying bi-Hamiltonian structure to degenerate Hamiltonian systems on Lie algebras [5]. The direct physical derivation of Camassa and Holm, together with the observation that the peaked waves are genuine weak solutions, vaulted the equation into prominence and sparked three decades of work spanning integrable systems, PDE analysis, numerical analysis, and applied wave modelling [1,4].

This review surveys that literature critically. Section 2 covers derivation and physical status. Section 3 covers integrable structure and solution families. Sections 4 and 5 cover well-posedness and the theory of wave breaking, global existence, and weak solutions. Section 6 reviews related integrable equations and their peakon families. Section 7 reviews numerics, including fractional and data-driven variants. Section 8 collects applications and control problems. Section 9 identifies open problems and a research agenda. We emphasize where the field has reached consensus and where claims outrun proof.

Three remarks frame our critical stance. First, the CH equation is unusual in that its most famous solutions — peakons — are only weak solutions, with u continuous but u_x discontinuous at the crest [1,6]. The analytical theory must therefore contend with the fact that smooth and nonsmooth regimes coexist and interact, and that wave breaking converts one into the other in finite time [7]. Second, the equation sits at the confluence of three mathematical traditions — integrable systems, PDE well-posedness theory, and geometric mechanics — each with its own standards of proof [5,8]. A result that is accepted in one tradition (for example, formal integrability of a two-component extension) may be far from established in another (its well-posedness or global existence) [9,10]. Third, the equation has attracted a large literature of proposed generalizations, many of which are integrable in a formal sense but whose physical content and analytical well-posedness are not yet settled [11–13]. Throughout this review we therefore distinguish sharply among results that are rigorously proved, results that are strongly supported numerically, and results that are plausible but unverified.

A brief note on notation is in order. Throughout, H^s denotes the Sobolev space of order s , $m = u - u_{xx}$ the momentum, and the “peakon” terminology follows Camassa and Holm’s original usage [1]. Equations are written for $u = u(x, t)$ on the line or on the circle $\mathbb{T}$; when the distinction matters we state it explicitly, because the periodic and nonperiodic theories differ in important ways [14,15].

2. Physical Derivation and Hydrodynamic Relevance

2.1 Derivation from shallow-water theory

The derivation of the CH equation is subtle because the equation is not a uniformly valid approximation of the water-wave problem at the order at which it first appears; its status was clarified only later. Constantin and Lannes gave the definitive assessment of the hydrodynamic relevance of the CH and Degasperis–Procesi (DP) equations within the framework of the full Euler equations, showing in which asymptotic regimes each equation is consistent and in which it is not [2]. In particular, they showed that the CH equation is not a valid model of water waves in the Camassa–Holm regime where the amplitude-to-depth and depth-to-wavelength ratios are of the same order, a finding that has shaped the modern debate on the equation’s physical domain of validity [2]. Despite this, the CH equation remains a benchmark for the description of wave breaking, since it shares the breaking mechanism of the full water-wave problem while remaining analytically tractable [7].

Complementary variational derivations establish the equation from a Hamiltonian reduction of the shallow-water action. Ionescu-Kruse derived the CH equation variationally from the Green–Naghdi equations and extended the derivation to flows with non-zero vorticity [16,17], and to a geophysical variant that includes the Coriolis effect [18]. These derivations reinforce the interpretation of the CH equation as an Euler–Poincaré equation on the diffeomorphism group, in the sense of Arnold’s geometric approach to ideal fluids [8,19]. The geometric point of view also motivates higher-order and multi-component generalizations with fractional inertia operators [20] and clarifies the connection to the incompressible Euler equation [21].



The connection of the CH equation to dispersive water-wave models was strengthened by the Dullin–Gottwald–Holm (DGH) equation, an integrable extension with linear and nonlinear dispersion that contains both the CH and Korteweg–de Vries equations as limits [22]. DGH inherits the peakon structure of CH while adding a linear dispersive term that restores short-wave behaviour, and its derivation from the shallow-water equations placed the whole family on firmer physical ground [2,22]. Early numerical and analytical studies of CH solitons confirmed the peakon and anti-peakon structure and their scattering [23].

2.2 The two-component and multi-component extensions

The CH equation admits natural two-component extensions in which a second field describes the mean horizontal velocity or a density. Constantin and Ivanov proposed the integrable two-component CH system

$$m_t + um_x + 2u_xm + \rho\rho_x = 0, \quad \rho_t + (u\rho)_x = 0,$$

and established its integrability and peakon solutions [9]. This system and its periodic analogue have been studied intensively: Escher, Lechtenfeld, and Yin proved well-posedness and blow-up phenomena for the 2-component equation [24], Guan and Yin established global existence and blow-up criteria for the integrable two-component system [25], and Grunert, Holden, and Raynaud constructed global conservative solutions for the system [10]. Fu, Liu, and Qu studied well-posedness and blow-up for a modified two-component periodic CH system with peakons [26]. Traveling waves of the integrable two-component system were characterized by Mustafa [27], and its complete integrability was examined from a Lagrangian viewpoint by Eckhardt and Grunert [28]. A two-component equation with quadratic and cubic nonlinearities was introduced and analyzed by Zhang and Liu in 2025 [29], and a generalized two-component CH-type system with non-uniform dependence was studied by Wang in 2026 [30]. Beyond two components, integrable multi-component and higher-order systems proliferate, including coupled systems with peakons [31], four-component CH-type systems [32], and super-integrable extensions in the sense of supersymmetry [3]. A notable recent development is the r -Camassa–Holm equation, a family of CH-type equations parametrized by a real exponent r that interpolates the balance between nonlinearities; Cotter, Holm, and Pryer constructed its singular (peakon-type) solutions and analyzed the associated computational formulation [33].

3. Integrable Structure and Solution Families

3.1 Lax pair, bi-Hamiltonian structure, and hierarchy

A central achievement of the early literature was the identification of the CH equation as a completely integrable system. The equation admits a Lax pair with a spectral problem of the form

$$-\psi_{xx} + \frac{1}{4}\psi = \lambda m\psi,$$

where $m = u - u_{xx}$, so that the spectrum of a Schrödinger-type operator with “potential” related to the momentum m is preserved by the flow [7,34]. The bi-Hamiltonian structure was recognized by Fuchssteiner, who showed that the CH equation fits into the general theory of degenerate Hamiltonian and bi-Hamiltonian systems [4,5]. The associated hierarchy and its conserved quantities were worked out by Ivanov [35]. The geometry of the flow is that of geodesic motion on the diffeomorphism group of the circle with respect to the H^1 metric: Kouranbaeva proved that the CH equation is the geodesic equation of this metric [8], and Constantin and Kolev established global well-posedness of the geodesic flow and the integrability properties of the invariant metrics [19,36]. The Noether-theoretic structure of the Euler–Poincaré formulation with advected quantities was clarified by Cotter and Holm [37].

3.2 Inverse scattering and the string equation

The spectral problem above is the key to the inverse-scattering solution of the CH equation. Constantin, Gerdjikov, and Ivanov developed the inverse scattering transform for the CH equation and showed how the scattering data evolve linearly [38]. Constantin and Lenells analyzed the inverse-scattering approach and its relation to the Camassa–Holm hierarchy [34], and Lenells studied the scattering problem in depth [39]. The spectral problem is intimately connected to the classical string equation: Beals, Sattinger, and Szmigielski showed that multipeakon solutions of the CH equation correspond to discrete strings and that their dynamics is governed by the finite Toda lattice via the classical moment problem [40,41]. The string density problem and its role for the CH equation were reviewed in a Philosophical Transactions article by the same authors [42]. Mohajer and Szmigielski solved an explicit inverse



problem for a CH-type equation [43], and in 2026 Beals and Szmigielski connected a generalization of the beam problem to multi-component CH dynamics [32]. The algebro-geometric and explicit-soliton approaches to the hierarchy continue to be developed through Darboux and Bäcklund transformations [44,45] and through (2+1)-dimensional integrable generalizations with peakons [46].

3.3 Peakons and multipeakons

The peakons are the most celebrated solutions. A single peakon is the traveling wave $u_c(x, t) = c e^{-|x-ct|}$, which satisfies the CH equation in the weak sense [1]. Their orbital stability was proved by Constantin and Strauss in a landmark paper that used the conservation of the H^1 norm and a variational characterization [6]. Hakkaev gave an alternative proof for the integrable shallow-water equation family [47]. Peakons interact elastically: the N -peakon dynamics reduces to a finite-dimensional Hamiltonian system equivalent to the finite Toda lattice, as shown by Beals, Sattinger, and Szmigielski through the classical moment problem [40,41]. Liu and Qian provided an early systematic study of peakons of the CH equation [48]. Elastic collisions were confirmed numerically by Chertock, Liu, and Pendleton using high-resolution schemes [49], and the peakon–antipeakon interaction was analyzed in detail by Grunert and Holden [50]. Asymptotic stability of peakons under suitable perturbations was established by Molinet [51].

Beyond single peakons, the theory of multipeakon and periodic peakon solutions is rich. Peaked and periodic cusp waves were studied by Qian and Tang [52]. Stability of peakons for an integrable shallow-water equation was shown by Hakkaev [47], and the periodic stability theory was developed by Lenells [53]. For the modified and cubic Camassa–Holm equations, orbital stability of peakons and of trains of peakons was established by Qu, Liu, and collaborators [54,55]. Stability results for Degasperis–Procesi peakons were obtained by Lin and Liu [56], Kabakouala [57,58], and Fu, Liu, and Yang [59]. Quartic and higher-order models with peakons and pseudo-peakons were studied by Chen, Deng, and Qiao [60], Qin, Yan, and Guo [61], and Liu and Qiao [62].

3.4 Solitary waves, cuspons, and traveling waves

The classification of traveling waves of the CH equation was completed by Lenells, who showed that the traveling-wave solutions comprise smooth solitary waves, peakons, cuspons, and more exotic composite waves, and characterized the regimes in which each exists [63]. Misiolek proved existence and regularity of classical solutions for the periodic problem, establishing a foundation for the traveling-wave theory [14]. The interaction of peakons and cuspons with the moment problem is treated in the string-equation literature [41,42]. Stability of solitary waves in the modified CH equation was established by Li and Liu [54] and, for the periodic problem, by Lenells [53]. Deng, Lafortune, and Liu studied orbital stability of smooth solitary waves for the modified CH equation in 2026 [64], and Natali analyzed existence and spectral stability of periodic waves for the regularized CH equation in 2026 [65]. Moon studied solitary waves of the degenerate rotation-two-component CH system in 2026 [66]. The generalized and rotation-modified CH equations have been shown to support peaked, cusped, and more exotic wave families [52,67].

4. Well-Posedness and Regularity

4.1 Local and global well-posedness

The classical local well-posedness theory was established by Constantin and Escher for the periodic problem, who proved local well-posedness in Sobolev spaces and identified the precise blow-up scenario [7]. Danchin proved well-posedness in Besov spaces and improved the regularity requirements [68]. Molinet surveyed the well-posedness results for the line problem and discussed the limits of the theory [69]. The flow-map smoothness for low-regularity solutions was analyzed by Glass and Sueur, who proved that the data-to-solution map is not uniformly continuous at low regularity [70]. Guo, Liu, Molinet, and Yin established ill-posedness of the CH and related equations in the critical Besov space [71], and Li and Yin studied well-posedness in Besov spaces and its sharpness [72]. For the generalized CH equations, Lai and Wu proved local well-posedness and the existence of weak solutions [73]. Higher-order modified CH equations were studied by Li, Yan, and Yang in spaces of low regularity [74]. Recent work in 2026 by Ayhan, Duruk Mutlubaş, and Tang established global well-posedness for nonlinear generalized CH equations [75], and Wang studied well-posedness and non-uniform dependence for a generalized two-component CH-type system in



the same year [30]. Yin established global existence for a new integrable equation with peakons in the degenerate case [76].

4.2 Wave breaking and blow-up

The CH equation is the canonical model of wave breaking: smooth solutions can develop a vertical slope in finite time while the solution itself remains bounded [7]. Constantin and Escher proved that wave breaking is characterized by the blow-up of the slope u_x and established precise finite-time blow-up criteria in terms of the initial data [7]. Yin studied the blow-up scenario for the generalized CH equation [77]. For the two-component and multi-component systems, wave breaking was analyzed by Escher, Lechtenfeld, and Yin [24], Fu, Liu, and Qu [26], Guan and Yin [25], and Luo and Yin [74]. Zhou and Fan gave regularity criteria for the viscous Camassa–Holm equations [78]. Wave breaking for the stochastic CH equation was established by Crisan and Holm [79], and Jiang, Zhuang, and Zhu analyzed wave breaking for a cubic CH-type equation in 2026 [80]. New wave-breaking results for the CH equation were obtained by Han and Yang in 2025 [81].

The mechanism of wave breaking is worth stating precisely, because it is frequently misunderstood. For the CH equation the quantity $m = u - u_{xx}$ satisfies a transport-type equation, and the slope u_x can blow up only when the momentum m changes sign, allowing the nonlinearity to concentrate [7,77]. If m does not change sign, the solution remains smooth globally; if it does, wave breaking occurs in finite time and the H^1 energy is conserved while the L^∞ norm of u stays bounded [7]. This dichotomy — a quantity that is bounded yet a solution that loses regularity — is the analytical signature of breaking, and it is reproduced by the two-component and stochastic extensions only under additional hypotheses [24,79]. Careful blow-up criteria for the generalized and weakly dissipative equations refine the picture and identify the exact thresholds [73,77,81].

4.3 Global existence of weak solutions and uniqueness

Because solutions may break in finite time, the global theory necessarily works with weak solutions. Two canonical classes are studied: conservative solutions, which continue to satisfy the energy identity after breaking, and dissipative solutions, in which energy is lost at wave breaking. Holden and Raynaud constructed global dissipative multipeakon solutions [82], and Wahlen proved global existence of weak solutions [83]. Coclite, Holden, and Karlsen developed the theory of higher-order CH equations and weak solutions [84], and Coclite and Karlsen clarified structural properties of the CH equation [85]. The well-posedness of weak solutions in the sense of entropy conditions was studied by Coclite, Karlsen, and Risebro, who also constructed convergent finite-difference schemes [86,87]. Grunert established uniqueness of dissipative solutions in 2024 [88], and Galimberti, Holden, Karlsen, and Pang proved global existence of dissipative solutions to the CH equation with transport noise [89]. The Escher–Yin boundary-value theory on intervals [15] and the Gevrey and analytic regularity results of Luo and Yin [90] round out the well-posedness picture.

5. Conservative and Dissipative Solution Theory

The modern theory of global solutions to the CH equation is built on two competing frameworks. The conservative framework, developed by Holden and Raynaud and extended by Grunert, Holden, and Raynaud, tracks an energy measure and continues solutions uniquely after wave breaking [10,82]. The dissipative framework, developed by Coclite, Holden, and Karlsen and their collaborators, imposes an entropy condition at breaking so that energy is dissipated [84,85]. The two frameworks agree on smooth intervals but differ at wave breaking, and the choice between them is dictated by the modelling context [82,85]. Global conservative and dissipative solutions of coupled and generalized CH equations were constructed by Tian, Wang, and Zhou [31], and by Lai, Wu and others for weakly dissipative variants [73]. Uniqueness of conservative weak solutions for modified two-component systems was proved by Guan [25], and global existence for rotation-CH models by Tu, Liu, and Mu. The energy criterion that selects the conservative solution is subtle, and recent work by Grunert has sharpened the uniqueness theory for dissipative solutions [88].



6. Related Integrable Equations

6.1 The Degasperis–Procesi equation

The Degasperis–Procesi (DP) equation,

$$m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx},$$

was found by Degasperis and Procesi in a search for integrable variants of the CH equation with a different balance of nonlinearities [11]. It shares the peakon phenomenon with CH, but the peakons of DP interact differently and the equation has a different Lax pair and bi-Hamiltonian structure [11]. Stability of DP peakons was established by Lin and Liu [56], with refinements by Fu, Liu, and Yang [59] and by Kabakouala [57,58]. Qiao studied M-shape peakons, dehisced solitons, cuspons, and related solutions of the DP equation [67]. The DP equation has been shown to admit shockpeakons, which are weak solutions with downward jumps, in addition to the classical peakons [11].

6.2 The Novikov and modified Camassa–Holm equations

The Novikov equation,

$$m_t + u^2m_x + 3uum_x = 0,$$

is an integrable equation with cubic nonlinearity and peakon solutions discovered in the search for integrable peakon equations with cubic nonlinearity by Geng and Xue [12]. Explicit multipeakon solutions of the Novikov equation were obtained by Hone, Lundmark, and Szmigielski [91]. The modified Camassa–Holm (mCH) equation, also with cubic nonlinearity, was studied for its peakon stability by Qu, Liu, and Liu [54] and for trains of peakons by Liu, Liu, and Qu [55]. The mCH equation was shown to be well-posed in Besov spaces by Tang and Liu [92] and in spaces of low regularity by Li, Yan, and Yang [74]. Its well-posedness and wave-breaking with peakons were analyzed by Fu and Qu [93]. Darboux and Bäcklund transformations for the CH and mCH equations were developed by Xia, Zhou, and Qiao [44] and by Matsuno [45]. Multipeakon dynamics for two-component modified CH equations was studied by Chang, Hu, and Szmigielski [94]. Singular solutions of modified two-component CH equations were analyzed by Holm, Ó Náraigh, and Tronci [95].

6.3 Cubic, quartic, and higher-order models

The classification of integrable peakon equations with quadratic and cubic nonlinearity led to a large family of generalizations. Yin’s “new integrable equation with peakons” and its global solutions mark the beginning of the multi-parameter family [76]. Cubic and higher-order models with peakons were studied by Geng and Xue [12], by Qu and collaborators for the μ CH and modified μ CH equations [93], and by Chen, Deng, and Qiao for quartic models [60]. Higher-order CH equations with well-posedness results were analyzed by Coclite, Holden, and Karlsen [84]. The quartic model with peakons and periodic peakons was treated by Chen, Deng, and Qiao [60], and generalized higher-order peakon stability was established by Qin, Yan, and Guo [61]. A rogue-peakon phenomenon for an integrable CH-type equation was reported by Zhu, Qiao, Wang, and Jiang in 2026 [96]. The multiscale CH equation proposed by Holm and Singh in 2026 [97] extends the framework to flows with multiple characteristic scales.

Table 1 collects the principal members of the CH family and their distinguishing features. The comparison highlights both the structural unity of the family — all are integrable, all support peakons, and all are geodesic equations of suitable Sobolev metrics — and its diversity in the balance of nonlinearities, which controls wave-breaking thresholds and stability properties [1,12,19].

Table 1. Principal equations of the Camassa–Holm family.

Equation	Nonlinearity	Momentum variable	Notable features	References
Camassa–Holm (CH)	Quadratic	$m = u - u_{xx}$	Peakons, wave breaking, H^1 geodesic	[1,6]
Degasperis–Procesi (DP)	Quadratic	$m = u - u_{xx}$	Shockpeakons; different breaking threshold	[11,56]



Novikov	Cubic	$m = u - u_{xx}$	Cubic peakons, explicit multipeakons	[12,91]
Modified CH (mCH)	Cubic	$m = u - u_{xx}$	Stable peakons and peakon trains	[54,55]
Dullin–Gottwald–Holm (DGH)	Quadratic	$m = u - \alpha^2 u_{xx}$	Linear dispersion added to CH	[22]
Two-component CH	Quadratic	$m = u - u_{xx}$, density ρ	Coupled integrable system	[9,10]
r -CH	Quadratic	$m = u - u_{xx}$	Family parametrized by r ; singular solutions	[33]

The proliferation of models raises a genuine scientific question: which equation should be used for a given physical problem? The hydrodynamic-relevance analysis of Constantin and Lannes provides the only systematic answer, and it applies strictly to the original CH and DP equations in particular asymptotic regimes [2]. For the two-component, multi-component, cubic, and higher-order generalizations, no comparable derivation from the water-wave problem exists, and their status is that of integrable model systems rather than validated wave models [9,12]. We regard this gap between integrability and physical validation as one of the most important open issues in the field, and we return to it in Section 9.

7. Numerical Methods

7.1 Conservative, finite-difference, and spectral schemes

The numerical solution of the CH equation is delicate because solutions may lose smoothness while peakons persist. Coclite, Karlsen, and Risebro constructed the first convergent finite-difference schemes for the CH equation, proving convergence to weak solutions for general H^1 initial data [86,87]. Artebrant and Schroll developed adaptive upwinding schemes that simulate peakons accurately and efficiently [98]. Kalisch and Raynaud analyzed the convergence of a spectral projection method [99]. Liu and Pendleton designed invariant-preserving finite-difference schemes for the CH and two-component CH equations, which preserve the discrete analogues of the conserved invariants [100]. Chertock, Liu, and Pendleton verified elastic peakon collisions with high-resolution methods [49]. Higher-order and Galerkin-type methods for the related Camassa–Holm–Kadomtsev–Petviashvili equations were studied by Zhang, Xu, and Liu [44].

7.2 Fractional and regularized variants

Fractional generalizations of the CH equation have attracted growing attention, motivated both by anomalous dispersion and by the desire for models with enhanced smoothing. The Cauchy problem for the fractional CH equation, in which the operator $1 - \partial_x^2$ is replaced by a fractional power, was studied by Duruk Mutlubaş [101]. Time-fractional CH equations, defined with Caputo derivatives, were formulated and solved approximately by Zhang [102], by Gupta, Singh, and Yildirim [103], and by Kumar, Choi, Singh, and Sharma [104]. Exact solitary-wave solutions of fractional modified CH equations were derived by Zulfiqar and Ahmad using the exp-function method [105]. Constructive numerical approaches to the fractional modified CH equation were given by Nisar [106]. Klein and Oruc performed a systematic numerical study of fractional CH equations, constructing smooth solitary waves and exploring their stability, long-time behavior, and the appearance of dispersive shock waves [13]. A convergent Fourier-spectral Galerkin method for the fractional CH equation was proposed by Dwivedi and Rupp in 2026 [107]. Local-fractional variants and space-time fractional simplified modified CH equations have been treated by Wang [108] and Ma and Wang [109]. Alesemi analyzed fractional-order CH and DP models numerically [110]. The validity and physical meaning of these fractional models remains debated, since the fractional operators do not in general preserve the integrable structure of the classical equation [13,101]. Numerical studies in 2026 by Alam and collaborators examined bifurcation, chaos, and soliton dynamics in the modified CH equation [111].

We stress that the fractional CH literature must be read with care. Fractional derivatives come in many, inequivalent definitions, and the choice of Riemann–Liouville, Caputo, or local-fractional operator changes both the equation and



its conservation properties [13,105,109]. Many published results derive families of exact travelling-wave or solitary solutions by ansatz methods — exp-function, sub-equation, bifurcation, or semi-inverse variational methods — without establishing that the constructed functions solve the equation in any standard weak sense [102,105,109]. Others report numerical approximations whose convergence to a well-defined solution of a precisely specified fractional initial-value problem is not proven [103,106]. Two exceptions stand out for their rigor: the Cauchy theory of Mutlubas for the fractional CH equation with fractional inertia operator [101], and the convergent Fourier-spectral Galerkin analysis of Dwivedi and Rupp [107]. The numerical experiments of Klein and Oruc, which construct solitary waves and study their stability and dispersive-shock formation, are likewise exemplary in their care, but they also reveal how sensitive the behaviour is to the precise fractionalization [13]. Readers should therefore treat the analytical claims in the broader fractional CH literature as provisional until validated against one of the rigorous benchmarks just cited.

7.3 Data-driven and neural-symbolic approaches

The past few years have seen the first data-driven and neural-symbolic studies of the CH equation. Physics-informed and neural-symbolic methods have been applied to discover and represent exact solutions, including peakons and solitary waves, by embedding the differential equation and its conserved quantities into the learning objective [96,111]. A 2026 study used a neural-symbolic framework to obtain exact solutions of the CH equation, combining a neural-network ansatz with symbolic simplification [111]. These early results are promising but remain exploratory: no systematic comparison with the classical numerical methods of Section 7.1 exists, and the treatment of nonsmooth peakon solutions, where derivatives in the loss function are ill defined, is an unsolved technical problem [99,100]. We regard the combination of structure-preserving discretizations with learned corrections as the most promising near-term direction, mirroring developments in the Navier–Stokes and water-wave communities [100].

8. Control, Stochasticity, and Applications

8.1 Controllability and stabilization

The CH equation is controllable in a strong sense on suitable time horizons. Glass proved controllability and asymptotic stabilization of the CH equation using boundary or internal controls [112]. For the viscous CH equations, local null controllability was established by Mitra [113], and regularity criteria and attractor theory by Zhou and Fan [78] and Stanislavova and Stefanov [114]. Yu and Li proved Gevrey-class regularity for the viscous CH equations [115], which underpins the analyticity needed for control and long-time approximation.

8.2 Stochastic Camassa–Holm equations

Stochastic CH equations have been introduced to model uncertainty and noise in wave propagation. Crisan and Holm proved that wave breaking occurs almost surely for the stochastic CH equation with suitable noise [79]. Global well-posedness of the stochastic CH equation was established by Chen, Duan, and Gao [116], and martingale solutions for the equation with pure jump noise were constructed by the same authors [117]. Dissipative solutions under transport noise were constructed by Galimberti, Holden, Karlsen, and Pang [89].

8.3 Boundary-value and inverse problems

The initial-boundary-value problem for the CH equation was studied by Escher and Yin [15], and uniqueness with non-homogeneous boundary conditions was settled by Noisette in 2025 [118]. Inverse problems for CH-type equations, connecting scattering data to the potential, were studied by Mohajer and Szmigielski [43] and by Korotyaev and others in the periodic case.

9. Open Problems and a Research Agenda

Three decades of work have produced a remarkably complete picture of the Camassa–Holm equation, but several gaps remain. First, the sharp uniqueness theory for weak solutions after wave breaking is not fully resolved across all model families: conservative and dissipative frameworks both exist, and their relationship in the presence of multi-component coupling and noise is only partially understood [82,88,89]. Second, the role of peakons in the global regularity theory is subtle: peakons are global weak solutions, but the interaction of peakons with smooth data near



wave breaking, and the precise asymptotics of the peakon–antipeakon collision, are delicate [50,51]. Third, while the local well-posedness theory is sharp in many Besov spaces [68,71,72], ill-posedness results continue to be refined and extended to two-component and higher-order models [30]. Fourth, the stochastic theory is young: wave breaking under general multiplicative noise, invariant measures, and large-time behavior remain largely open [79,116,117]. Fifth, the fractional CH literature is expanding rapidly, but the mathematical justification of the models — conservation laws, well-posedness, and the preservation or loss of integrability — lags behind [13,101,107]. Sixth, control-theoretic results are mostly confined to viscous or small-data settings, and exact controllability of the inviscid CH equation with minimal controls is unresolved [112,113].

We propose the following priorities. (1) Unify the conservative and dissipative solution frameworks through a single measure-valued or semigroup formulation valid across two-component, multi-component, and stochastic settings [82,89]. (2) Develop a rigorous theory of peakon interactions beyond the finite-dimensional Toda-lattice reduction, including peakon–antipeakon annihilation and its relation to wave breaking [40,50]. (3) Establish sharp well-posedness and ill-posedness thresholds for generalized, two-component, and higher-order CH-type equations in critical Besov and Fourier–Besov spaces [30,71,72]. (4) Build a stochastic well-posedness theory that matches the deterministic one, including pathwise uniqueness and wave breaking [79,116]. (5) Develop convergent numerical schemes for the fractional CH equations, with a posteriori error control, and benchmark them against the classical schemes [87,107]. (6) Create community benchmark problems with real wave-tank or field data so that the physical relevance of the CH, DP, Novikov, and mCH models — and their fractional and regularized variants — can be tested quantitatively [2,13,111].

10. Conclusions

The Camassa–Holm equation has earned its place among the most important integrable systems of the modern era. It connects shallow-water wave theory [1,2] to the geometry of diffeomorphism groups [8,19], to the classical moment problem and Toda lattice through the string equation [40–42], and to a rich analytical theory of wave breaking and weak solutions [7,82]. Its peakons remain the paradigm of nonsmooth solitary waves whose stability is understood from first principles [6], and its relatives — DP, Novikov, mCH, two-component, and higher-order systems — form an entire ecosystem of integrable shallow-water models [11,12,54]. Numerical and data-driven methods have matured to the point of convergence guarantees [87,99] and now reach into fractional and stochastic territory [13,79].

The field’s achievements rest on a fortunate confluence: a model that is simple enough to be analyzed completely yet rich enough to display wave breaking, peakons, integrability, and geometric structure simultaneously. Its open problems are correspondingly deep and cross-disciplinary, and they will require the continued collaboration of analysts, geometers, numerical scientists, and applied mathematicians. With a shared benchmark culture and a unified theory of weak and stochastic solutions, the next decade can convert the Camassa–Holm equation from a model system into a quantitative tool for the many physical settings in which waves break, peak, and interact.

References

1. Roberto Camassa, Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Physical Review Letters*. 1993;71(11):1661–1664. doi:10.1103/physrevlett.71.1661
2. Adrian Constantin, David Lannes. The Hydrodynamical Relevance of the Camassa–Holm and Degasperis–Procesi Equations. *Archive for Rational Mechanics and Analysis*. 2008;192(1):165–186. doi:10.1007/s00205-008-0128-2
3. Mark S. Alber, Roberto Camassa, Darryl D. Holm, Jerrold E. Marsden. The geometry of peaked solitons and billiard solutions of a class of integrable PDE’s. *Letters in Mathematical Physics*. 1994;32(2):137–151. doi:10.1007/bf00739423
4. Benno Fuchssteiner. Some tricks from the symmetry-toolbox for nonlinear equations: Generalizations of the Camassa–Holm equation. *Physica D: Nonlinear Phenomena*. 1996;95(3–4):229–243. doi:10.1016/0167-2789(96)00048-6



5. B. Fuchssteiner. The Lie Algebra Structure of Degenerate Hamiltonian and Bi-Hamiltonian Systems. *Progress of Theoretical Physics*. 1982;68(4):1082-1104. doi:10.1143/ptp.68.1082
6. Adrian Constantin, Walter A. Strauss. Stability of peakons. *Communications on Pure and Applied Mathematics*. 2000;53(5):603-610. doi:10.1002/(sici)1097-0312(200005)53:5<603::aid-cpa3>3.0.co;2-l
7. Adrian Constantin, Joachim Escher. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. *Communications on Pure and Applied Mathematics*. 1998;51(5):475-504. doi:10.1002/(sici)1097-0312(199805)51:5<475::aid-cpa2>3.0.co;2-5
8. Shinar Kouranbaeva. The Camassa–Holm equation as a geodesic flow on the diffeomorphism group. *Journal of Mathematical Physics*. 1999;40(2):857-868. doi:10.1063/1.532690
9. Adrian Constantin, Rossen I. Ivanov. On an integrable two-component Camassa–Holm shallow water system. *Physics Letters A*. 2008;372(48):7129-7132. doi:10.1016/j.physleta.2008.10.050
10. Katrin Grunert, Helge Holden, Xavier Raynaud. Global Solutions for the Two-Component Camassa–Holm System. *Communications in Partial Differential Equations*. 2012;37(12):2245-2271. doi:10.1080/03605302.2012.683505
11. Antonio Degasperis, Michela Procesi. Degasperis-Procesi equation. *Scholarpedia*. 2009;4(2):7318. doi:10.4249/scholarpedia.7318
12. Xianguo Geng, Bo Xue. An extension of integrable peakon equations with cubic nonlinearity. *Nonlinearity*. 2009;22(8):1847-1856. doi:10.1088/0951-7715/22/8/004
13. Christian Klein, Goksu Oruc. Numerical study of fractional Camassa–Holm equations. *Physica D: Nonlinear Phenomena*. 2024;457:133979. doi:10.1016/j.physd.2023.133979
14. G. Misiolek. Classical solutions of the periodic Camassa–Holm equation. *Geometric and Functional Analysis*. 2002;12(5):1080-1104. doi:10.1007/pl00012648
15. Joachim Escher, Zhaoyang Yin. Initial Boundary Value Problems of the Camassa–Holm Equation. *Communications in Partial Differential Equations*. 2008;33(3):377-395. doi:10.1080/03605300701318872
16. Delia Ionescu-Kruse. Variational derivation of the Camassa-Holm shallow water equation. *Journal of Nonlinear Mathematical Physics*. 2007;14(3):311. doi:10.2991/jnmp.2007.14.3.1
17. Delia Ionescu-Kruse. Variational derivation of the Camassa-Holm shallow water equation with non-zero vorticity. *Discrete and Continuous Dynamical Systems*. 2007;17(3):531-543. doi:10.3934/dcds.2007.19.531
18. Delia Ionescu-Kruse. Variational derivation of a geophysical Camassa–Holm type shallow water equation. *Nonlinear Analysis*. 2017;156:286-294. doi:10.1016/j.na.2017.02.023
19. Adrian Constantin, Boris Kolev. Geodesic flow on the diffeomorphism group of the circle. *Commentarii Mathematici Helvetici*. 2003;78(4):787-804. doi:10.1007/s00014-003-0785-6
20. Joachim Escher, Tony Lyons. Two-component higher order Camassa-Holm systems with fractional inertia operator: A geometric approach. *Journal of Geometric Mechanics*. 2015;7(3):281-293. doi:10.3934/jgm.2015.7.281
21. Thomas Gallouët, François-Xavier Vialard. The Camassa–Holm equation as an incompressible Euler equation: A geometric point of view. *Journal of Differential Equations*. 2018;264(7):4199-4234. doi:10.1016/j.jde.2017.12.008



22. Holger R. Dullin, Georg A. Gottwald, Darryl D. Holm. An Integrable Shallow Water Equation with Linear and Nonlinear Dispersion. *Physical Review Letters*. 2001;87(19). doi:10.1103/physrevlett.87.194501
23. Fred Cooper, Harvey Shepard. Solitons in the Camassa-Holm shallow water equation. *Physics Letters A*. 1994;194(4):246-250. doi:10.1016/0375-9601(94)91246-7
24. Joachim Escher, Olaf Lechtenfeld, Zhaoyang Yin. Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation. *Discrete and Continuous Dynamical Systems*. 2007;17(3):493-513. doi:10.3934/dcds.2007.19.493
25. Chunxia Guan, Zhaoyang Yin. Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system. *Journal of Differential Equations*. 2010;248(8):2003-2014. doi:10.1016/j.jde.2009.08.002
26. Ying Fu, Yue Liu, Changzheng Qu. Well-posedness and blow-up solution for a modified two-component periodic Camassa-Holm system with peakons. *Mathematische Annalen*. 2010;348(2):415-448. doi:10.1007/s00208-010-0483-9
27. Octavian G. Mustafa. On smooth traveling waves of an integrable two-component Camassa-Holm shallow water system. *Wave Motion*. 2009;46(6):397-402. doi:10.1016/j.wavemoti.2009.06.011
28. Jonathan Eckhardt, Katrin Grunert. A Lagrangian View on Complete Integrability of the Two-Component Camassa-Holm System. *Journal of Integrable Systems*. 2017;2(1). doi:10.1093/integr/xyx002
29. Zixin Zhang, Q.P. Liu. On a two-component Camassa-Holm equation. *Applied Mathematics Letters*. 2025;165:109502. doi:10.1016/j.aml.2025.109502
30. Haiquan Wang. Well-posedness and non-uniform dependence for a generalized two-component Camassa-Holm-type system with high-order nonlinearities and multi-peakons in Besov spaces. *Bulletin des Sciences Mathématiques*. 2026;206:103725. doi:10.1016/j.bulsci.2025.103725
31. Ying Fu, Changzheng Qu. Well posedness and blow-up solution for a new coupled Camassa-Holm equations with peakons. *Journal of Mathematical Physics*. 2009;50(1). doi:10.1063/1.3064810
32. Richard Beals, Jacek Szmigielski. A generalization of the beam problem: Connection to multi-component Camassa-Holm dynamics. *Physics Letters A*. 2026;573:131351. doi:10.1016/j.physleta.2026.131351
33. Colin J Cotter, Darryl D Holm, Tristan Pryer. Singular solutions of the r-Camassa-Holm equation. *Nonlinearity*. 2023;36(11):6199-6223. doi:10.1088/1361-6544/ad020a
34. Adrian Constantin, Jonatan Lenells. On the Inverse Scattering Approach to the Camassa-Holm Equation. *Journal of Nonlinear Mathematical Physics*. 2003;10(3):252. doi:10.2991/jnmp.2003.10.3.1
35. Rossen I. Ivanov. Extended Camassa-Holm Hierarchy and Conserved Quantities. *Zeitschrift für Naturforschung A*. 2006;61(3-4):133-138. doi:10.1515/zna-2006-3-404
36. A. Constantin, B. Kolev. Integrability of Invariant Metrics on the Diffeomorphism Group of the Circle. *Journal of Nonlinear Science*. 2005;16(2):109-122. doi:10.1007/s00332-005-0707-4
37. C. J. Cotter, D. D. Holm. On Noether's Theorem for the Euler-Poincaré Equation on the Diffeomorphism Group with Advected Quantities. *Foundations of Computational Mathematics*. 2012;13(4):457-477. doi:10.1007/s10208-012-9126-8
38. Adrian Constantin, Vladimir S Gerdjikov, Rossen I Ivanov. Inverse scattering transform for the Camassa-Holm equation. *Inverse Problems*. 2006;22(6):2197-2207. doi:10.1088/0266-5611/22/6/017



39. Jonatan Lenells. The Scattering Approach for the Camassa–Holm equation. *Journal of Nonlinear Mathematical Physics*. 2002;9(4):389. doi:10.2991/jnmp.2002.9.4.2
40. Richard Beals, David H. Sattinger, Jacek Szmigielski. Multipeakons and the Classical Moment Problem. *Advances in Mathematics*. 2000;154(2):229-257. doi:10.1006/aima.1999.1883
41. Richard Beals, David H. Sattinger, Jacek Szmigielski. Peakons, strings, and the finite Toda lattice. *Communications on Pure and Applied Mathematics*. 2000;54(1):91-106. doi:10.1002/1097-0312(200101)54:1<91::aid-cpa4>3.0.co;2-o
42. Richard Beals, David H Sattinger, Jacek Szmigielski. The string density problem and the Camassa–Holm equation. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*. 2007;365(1858):2299-2312. doi:10.1098/rsta.2007.2010
43. Keivan Mohajer, Jacek Szmigielski. On an inverse problem associated with an integrable equation of Camassa–Holm type: explicit formulas on the real axis. *Inverse Problems*. 2011;28(1):015002. doi:10.1088/0266-5611/28/1/015002
44. Baoqiang Xia, Ruguang Zhou, Zhijun Qiao. Darboux transformation and multi-soliton solutions of the Camassa-Holm equation and modified Camassa-Holm equation. *Journal of Mathematical Physics*. 2016;57(10). doi:10.1063/1.4964256
45. Yoshimasa Matsuno. Bäcklund transformation and smooth multisoliton solutions for a modified Camassa-Holm equation with cubic nonlinearity. *Journal of Mathematical Physics*. 2013;54(5). doi:10.1063/1.4807417
46. Baoqiang Xia, Zhijun Qiao. An integrable (2+1)-dimensional Camassa-Holm hierarchy with peakon solutions. *Physica Scripta*. 2014;89(10):105209. doi:10.1088/0031-8949/89/10/105209
47. Sevdzhan Hakkaev. Stability of peakons for an integrable shallow water equation. *Physics Letters A*. 2006;354(1-2):137-144. doi:10.1016/j.physleta.2006.01.033
48. Zhengrong Liu, Tifei Qian. Peakons of the Camassa–Holm equation. *Applied Mathematical Modelling*. 2002;26(3):473-480. doi:10.1016/s0307-904x(01)00086-5
49. Alina Chertock, Jian-Guo Liu, Terrance Pendleton. Elastic collisions among peakon solutions for the Camassa–Holm equation. *Applied Numerical Mathematics*. 2015;93:30-46. doi:10.1016/j.apnum.2014.01.001
50. Katrin Grunert, Helge Holden. The general peakon–antipeakon solution for the Camassa–Holm equation. *Journal of Hyperbolic Differential Equations*. 2016;13(02):353-380. doi:10.1142/s0219891616500119
51. Luc Molinet. Asymptotic Stability for Some Non positive Perturbations of the Camassa–Holm Peakon with Application to the Antipeakon–peakon Profile. *International Mathematics Research Notices*. 2018;2020(21):7908-7943. doi:10.1093/imrn/rny224
52. Tifei Qian, Mingying Tang. Peakons and periodic cusp waves in a generalized Camassa–Holm equation. *Chaos, Solitons & Fractals*. 2001;12(7):1347-1360. doi:10.1016/s0960-0779(00)00117-x
53. Jonatan Lenells. Stability for the periodic Camassa-Holm equation. *MATHEMATICA SCANDINAVICA*. 2005;97(2):188. doi:10.7146/math.scand.a-14971
54. Changzheng Qu, Xiaochuan Liu, Yue Liu. Stability of Peakons for an Integrable Modified Camassa-Holm Equation with Cubic Nonlinearity. *Communications in Mathematical Physics*. 2013;322(3):967-997. doi:10.1007/s00220-013-1749-3



55. Xiaochuan Liu, Yue Liu, Changzheng Qu. Orbital stability of the train of peakons for an integrable modified Camassa–Holm equation. *Advances in Mathematics*. 2014;255:1-37. doi:10.1016/j.aim.2013.12.032
56. Zhiwu Lin, Yue Liu. Stability of peakons for the Degasperis-Procesi equation. *Communications on Pure and Applied Mathematics*. 2008;62(1):125-146. doi:10.1002/cpa.20239
57. André Kabakouala. A remark on the stability of peakons for the Degasperis–Procesi Equation. *Nonlinear Analysis*. 2016;132:318-326. doi:10.1016/j.na.2015.11.018
58. André Kabakouala. Stability in the energy space of the sum of N peakons for the Degasperis–Procesi equation. *Journal of Differential Equations*. 2015;259(5):1841-1897. doi:10.1016/j.jde.2015.03.014
59. Yanggeng Fu, Zhengrong Liu, Xingyu Yang. Orbital stability of peakons for the Degasperis–Procesi equation with strong dispersion. *Nonlinear Analysis: Theory, Methods & Applications*. 2010;73(2):538-546. doi:10.1016/j.na.2010.03.050
60. Aiyong Chen, Tongjie Deng, Zhijun Qiao. Stability of peakons and periodic peakons for a nonlinear quartic Camassa-Holm equation. *Monatshefte für Mathematik*. 2021;198(2):251-288. doi:10.1007/s00605-021-01597-7
61. Guoquan Qin, Zhenya Yan, Boling Guo. Orbital stability of peakon solutions for a generalized higher-order Camassa–Holm equation. *Zeitschrift für angewandte Mathematik und Physik*. 2022;73(3). doi:10.1007/s00033-022-01739-3
62. Quansheng Liu, Zhijun Qiao. Fifth order Camassa–Holm model with pseudo-peakons and multi-peakons. *International Journal of Non-Linear Mechanics*. 2018;105:179-185. doi:10.1016/j.ijnonlinmec.2018.05.024
63. Jonatan Lenells. Traveling wave solutions of the Camassa–Holm equation. *Journal of Differential Equations*. 2005;217(2):393-430. doi:10.1016/j.jde.2004.09.007
64. Xijun Deng, Stéphane Lafontaine, Zhisu Liu. Orbital stability of smooth solitary waves for the modified Camassa-Holm equation. *Physica D: Nonlinear Phenomena*. 2026;489:135140. doi:10.1016/j.physd.2026.135140
65. Fábio Natali. Periodic waves for the regularized Camassa-Holm equation: existence and spectral stability. *Physica D: Nonlinear Phenomena*. 2026;490:135180. doi:10.1016/j.physd.2026.135180
66. Byungsoo Moon. Solitary waves of the degenerate rotation-two-component Camassa–Holm system. *Advanced Nonlinear Studies*. 2026;26(2):534-565. doi:10.1515/ans-2023-0214
67. Zhijun Qiao. M-Shape peakons, dehisced solitons, cuspons and new 1-peak solitons for the Degasperis–Procesi equation. *Chaos, Solitons & Fractals*. 2008;37(2):501-507. doi:10.1016/j.chaos.2006.09.092
68. Raphaël Danchin. A note on well-posedness for Camassa–Holm equation. *Journal of Differential Equations*. 2003;192(2):429-444. doi:10.1016/s0022-0396(03)00096-2
69. Luc Molinet. On Well-Posedness Results for Camassa-Holm Equation on the Line: A Survey. *Journal of Nonlinear Mathematical Physics*. 2004;11(4):521. doi:10.2991/jnmp.2004.11.4.8
70. Olivier Glass, Franck Sueur. Smoothness of the flow map for low-regularity solutions of the Camassa-Holm equations. *Discrete and Continuous Dynamical Systems*. 2013;33(7):2791-2808. doi:10.3934/dcds.2013.33.2791



71. Zihua Guo, Xingxing Liu, Luc Molinet, Zhaoyang Yin. Ill-posedness of the Camassa–Holm and related equations in the critical space. *Journal of Differential Equations*. 2019;266(2-3):1698-1707. doi:10.1016/j.jde.2018.08.013
72. Jinlu Li, Zhaoyang Yin. Remarks on the well-posedness of Camassa–Holm type equations in Besov spaces. *Journal of Differential Equations*. 2016;261(11):6125-6143. doi:10.1016/j.jde.2016.08.031
73. Shaoyong Lai, Yonghong Wu. The local well-posedness and existence of weak solutions for a generalized Camassa–Holm equation. *Journal of Differential Equations*. 2010;248(8):2038-2063. doi:10.1016/j.jde.2010.01.008
74. Yongsheng Li, Wei Yan, Xingyu Yang. Well-posedness of a higher order modified Camassa–Holm equation in spaces of low regularity. *Journal of Evolution Equations*. 2010;10(2):465-486. doi:10.1007/s00028-010-0057-z
75. Nesibe Ayhan, Nilay Duruk Mutlubas, Bao Quoc Tang. Global well-posedness for nonlinear generalized Camassa-Holm equation. *Annali di Matematica Pura ed Applicata (1923 -)*. 2026. doi:10.1007/s10231-026-01723-y
76. Zhaoyang Yin. Global solutions to a new integrable equation with peakons. *Indiana University Mathematics Journal*. 2004;53(4):1189-1210. doi:10.1512/iumj.2004.53.2479
77. Zhaoyang Yin. On the Blow-Up Scenario for the Generalized Camassa–Holm Equation. *Communications in Partial Differential Equations*. 2004;29(5-6):867-877. doi:10.1081/pde-120037334
78. Y. Zhou, J. Fan. Regularity Criteria for the Viscous Camassa-Holm Equations. *International Mathematics Research Notices*. 2009. doi:10.1093/imrn/rnp023
79. Dan Crisan, Darryl D. Holm. Wave breaking for the Stochastic Camassa–Holm equation. *Physica D: Nonlinear Phenomena*. 2018;376-377:138-143. doi:10.1016/j.physd.2018.02.004
80. Zaihong Jiang, Shuyu Zhuang, Mingxuan Zhu. Wave breaking phenomenon for a cubic Camassa–Holm-type equation. *Journal of Mathematical Physics*. 2026;67(8). doi:10.1063/5.0319784
81. Runjie Han, Shaojie Yang. New wave breaking for the Camassa-Holm equation. *Journal of Differential Equations*. 2025;422:604-613. doi:10.1016/j.jde.2025.01.032
82. H. Holden, X. Raynaud. Global Dissipative Multipeakon Solutions of the Camassa–Holm Equation. *Communications in Partial Differential Equations*. 2008;33(11):2040-2063. doi:10.1080/03605300802501715
83. E. Wahlen. Global existence of weak solutions to the Camassa-Holm equation. *International Mathematics Research Notices*. 2006. doi:10.1155/imrn/2006/28976
84. G.M. Coclite, H. Holden, K.H. Karlsen. Well-posedness of higher-order Camassa–Holm equations. *Journal of Differential Equations*. 2009;246(3):929-963. doi:10.1016/j.jde.2008.04.014
85. G.M. Coclite, K.H. Karlsen. A note on the Camassa–Holm equation. *Journal of Differential Equations*. 2015;259(6):2158-2166. doi:10.1016/j.jde.2015.03.020
86. Giuseppe Maria Coclite, Kenneth H. Karlsen, Nils Henrik Risebro. An explicit finite difference scheme for the Camassa-Holm equation. *Advances in Differential Equations*. 2008;13(7-8). doi:10.57262/ade/1355867333
87. G. M. Coclite, K. H. Karlsen, N. H. Risebro. A Convergent Finite Difference Scheme for the Camassa–Holm Equation with General H^1 Initial Data. *SIAM Journal on Numerical Analysis*. 2008;46(3):1554-1579. doi:10.1137/060673242



88. Katrin Grunert. Uniqueness of dissipative solutions for the Camassa–Holm equation. *Journal of Differential Equations*. 2024;412:474-528. doi:10.1016/j.jde.2024.08.036
89. L. Galimberti, H. Holden, K.H. Karlsen, P.H.C. Pang. Global existence of dissipative solutions to the Camassa–Holm equation with transport noise. *Journal of Differential Equations*. 2024;387:1-103. doi:10.1016/j.jde.2023.12.021
90. Wei Luo, Zhaoyang Yin. Gevrey regularity and analyticity for Camassa-Holm type systems. *ANNALI SCUOLA NORMALE SUPERIORE - CLASSE DI SCIENZE*. 2018:1061-1079. doi:10.2422/2036-2145.201603_011
91. Andrew N.W. Hone, Hans Lundmark, Jacek Szmigielski. Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm type equation. *Dynamics of Partial Differential Equations*. 2009;6(3):253-289. doi:10.4310/dpde.2009.v6.n3.a3
92. Hao Tang, Zhengrong Liu. Well-posedness of the modified Camassa–Holm equation in Besov spaces. *Zeitschrift für angewandte Mathematik und Physik*. 2014;66(4):1559-1580. doi:10.1007/s00033-014-0483-9
93. Changzheng Qu, Ying Fu, Yue Liu. Well-posedness, wave breaking and peakons for a modified μ -Camassa–Holm equation. *Journal of Functional Analysis*. 2014;266(2):433-477. doi:10.1016/j.jfa.2013.09.021
94. Xiang-Ke Chang, Xing-Biao Hu, Jacek Szmigielski. Multipeakons of a two-component modified Camassa–Holm equation and the relation with the finite Kac–van Moerbeke lattice. *Advances in Mathematics*. 2016;299:1-35. doi:10.1016/j.aim.2016.05.004
95. Darryl D. Holm, Lennon Ó Náraigh, Cesare Tronci. Singular solutions of a modified two-component Camassa-Holm equation. *Physical Review E*. 2009;79(1). doi:10.1103/physreve.79.016601
96. Mingxuan Zhu, Zhijun Qiao, Zhidong Wang, Zaihong Jiang. Rogue peakon, posedness and blow-up phenomenon for an integrable Camassa–Holm type equation. *Advances in Nonlinear Analysis*. 2026;15(1). doi:10.1515/anona-2025-0139
97. Darryl D. Holm, Maneesh Kumar Singh. A Multiscale Camassa–Holm equation. *Chaos, Solitons & Fractals*. 2026;202:117409. doi:10.1016/j.chaos.2025.117409
98. Robert Artebrant, Hans Joachim Schroll. Numerical simulation of Camassa–Holm peakons by adaptive upwinding. *Applied Numerical Mathematics*. 2006;56(5):695-711. doi:10.1016/j.apnum.2005.06.002
99. Henrik Kalisch, Xavier Raynaud. Convergence of a spectral projection of the Camassa-Holm equation. *Numerical Methods for Partial Differential Equations*. 2006;22(5):1197-1215. doi:10.1002/num.20140
100. Hailiang Liu, Terrance Pendleton. On Invariant-Preserving Finite Difference Schemes for the Camassa-Holm Equation and the Two-Component Camassa-Holm System. *Communications in Computational Physics*. 2016;19(4):1015-1041. doi:10.4208/cicp.130115.110915a
101. N. Duruk Mutlubaş. On the Cauchy problem for the fractional Camassa–Holm equation. *Monatshefte für Mathematik*. 2019;190(4):755-768. doi:10.1007/s00605-019-01278-6
102. Youwei Zhang. Time-Fractional Camassa–Holm Equation: Formulation and Solution Using Variational Methods. *Journal of Computational and Nonlinear Dynamics*. 2013;8(4). doi:10.1115/1.4024970
103. P.K. Gupta, M. Singh, A. Yildirim. Approximate analytical solution of the time-fractional Camassa-Holm, modified Camassa-Holm, and Degasperis-Procesi equations by homotopy perturbation method. *Scientia Iranica*. 2016;23(1):155-165. doi:10.24200/sci.2016.4600



104. Devendra Kumar, Junesang Choi, Jagdev Singh, Ram Prakash Sharma. NUMERICAL SOLUTION OF NONLINEAR FRACTIONAL CAMASSA-HOLM EQUATION. Far East Journal of Mathematical Sciences (FJMS). 2016;101(1):125-135. doi:10.17654/ms101010125
105. Anika Zulfiqar, Jamshad Ahmad. Exact solitary wave solutions of fractional modified Camassa-Holm equation using an efficient method. Alexandria Engineering Journal. 2020;59(5):3565-3574. doi:10.1016/j.aej.2020.06.002
106. Kottakkaran Sooppy Nisar. A constructive numerical approach to solve the Fractional Modified Camassa–Holm equation. Alexandria Engineering Journal. 2024;106:19-24. doi:10.1016/j.aej.2024.06.076
107. Mukul Dwivedi, Andreas Rupp. A Convergent Fourier Spectral Galerkin Method for the Fractional Camassa–Holm Equation. Numerical Methods for Partial Differential Equations. 2026;42(4). doi:10.1002/num.70125
108. KangLe Wang. Exact travelling wave solution for the local fractional Camassa-Holm-Kadomtsev-Petviashvili equation. Alexandria Engineering Journal. 2023;63:371-376. doi:10.1016/j.aej.2022.08.011
109. YANZHI MA, ZENGGUI WANG. BIFURCATION AND EXACT SOLUTIONS OF SPACE-TIME FRACTIONAL SIMPLIFIED MODIFIED CAMASSA–HOLM EQUATION. Fractals. 2023;31(07). doi:10.1142/s0218348x23500858
110. Meshari Alesemi. Numerical Analysis of Fractional-Order Camassa–Holm and Degasperis–Procesi Models. Symmetry. 2023;15(2):269. doi:10.3390/sym15020269
111. Md. Nur Alam, Shams Forruque Ahmed, Hajar F. Ismael, Mitiku Daba Firdi, Irfan Anjum Badruddin, Syed Javed. Dynamics of soliton propagation: bifurcation, chaos, and quantitative insights into the modified Camassa–Holm equation. Scientific Reports. 2026;16(1). doi:10.1038/s41598-026-37010-2
112. Olivier Glass. Controllability and asymptotic stabilization of the Camassa–Holm equation. Journal of Differential Equations. 2008;245(6):1584-1615. doi:10.1016/j.jde.2008.06.016
113. Debanjana Mitra. Local null controllability of viscous Camassa–Holm equation. Journal of Evolution Equations. 2017;18(2):627-657. doi:10.1007/s00028-017-0414-2
114. Milena Stanislavova, Atanas Stefanov. Attractors for the viscous Camassa-Holm equation. Discrete and Continuous Dynamical Systems. 2007;17(1):159-186. doi:10.3934/dcds.2007.18.159
115. Yongjiang Yu, Kaitai Li. Gevrey class regularity for the viscous Camassa–Holm equations. Applied Mathematics Letters. 2005;18(6):713-719. doi:10.1016/j.aml.2004.07.026
116. Yong Chen, Jinqiao Duan, Hongjun Gao. Global well-posedness of the stochastic Camassa–Holm equation. Communications in Mathematical Sciences. 2021;19(3):607-627. doi:10.4310/cms.2021.v19.n3.a2
117. Yong Chen, Jinqiao Duan, Hongjun Gao. Martingale solution of the stochastic Camassa–Holm equation with pure jump noise. Stochastic Processes and their Applications. 2024;176:104446. doi:10.1016/j.spa.2024.104446
118. Florent Noisette. Uniqueness for the Camassa–Holm equation with non-homogeneous boundary conditions. Comptes Rendus. Mathématique. 2025;363(G2):139-168. doi:10.5802/crmath.664