



# The Estevez–Prada Equation: A Comprehensive Review with a Focus on High-Dimensional Wave Dynamics

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## Abstract

Nonlinear evolution equations in three spatial dimensions underpin the modelling of wave propagation in plasmas, optical fibres, oceans, and complex fluids. The Estevez–Prada equation (EPE), introduced by Estévez and Prada in the early 2000s as a higher-dimensional extension of integrable systems, has recently re-emerged as a versatile model of anisotropic, high-dimensional wave dynamics. This review provides a comprehensive and critical synthesis of the EPE literature up to 2026, with an explicit focus on its (3+1)-dimensional form. We describe the mathematical origins of the equation, examine its Painlevé integrability, and organise the extensive family of analytical methods that have been applied to it—ranging from the Hirota bilinear method, Darboux and Bäcklund transformations, and Lie symmetry analysis to the tanh-coth, (G/G)-expansion, Jacobi elliptic function, and modified extended direct algebraic methods. We then classify the wave structures that these methods generate, including kinks, anti-kinks, dark and bright solitons, singular and periodic-singular waves, kink–antikink interactions, lumps, rogue waves, and breathers. A critical analysis identifies recurring methodological weaknesses, including solution redundancy, limited physical validation, and inconsistent verification, and documents clear trends toward variable-coefficient, fractional, stochastic, and data-driven formulations. A comparative synthesis of 133 verified primary sources is presented, and open problems and future research directions are highlighted. The review concludes that the EPE is a

mathematically rich and physically relevant benchmark for high-dimensional nonlinear wave theory, but that its quantitative validation and standardisation remain incomplete.

**Keywords:** Estevez–Prada equation; high-dimensional nonlinear systems; Painlevé integrability; solitons; exact solutions; fractional and stochastic models; physics-informed neural networks

## 1. Introduction

Nonlinear partial differential equations (NLPDEs) are the principal language of continuum physics. They describe wave propagation, fluid motion, heat transfer, electromagnetic radiation, and quantum matter, and they become indispensable whenever linear superposition fails [1,2]. Many of the most consequential phenomena of modern science—stable localised waves, coherent structures, and pattern formation—emerge from an interplay between dispersion and nonlinearity that no linear model can reproduce [3,4]. As physical descriptions grow more realistic, they also grow more demanding: real media are anisotropic, inhomogeneous, and high-dimensional, and the corresponding equations rarely admit closed-form solutions [5,6].

Among the many integrable models that capture such behaviour, the Estevez–Prada equation (EPE) occupies a distinctive place. It was introduced by Pilar G. Estévez and Julia Prada in a programme that extended classical integrable equations from one and two dimensions into richer geometries [7]. Their work generalised the sine–Gordon equation to 2+1 dimensions [7], developed the singular manifold method for equations in 2+1 dimensions [8], and constructed integrable hierarchies and hodograph transformations for Camassa–Holm-type systems [9,10]. This body



of work established the EPE and its relatives as natural testing grounds for questions of integrability, symmetry, and explicit solvability in higher dimensions [11,12].

For nearly two decades the EPE remained a specialist topic. Interest has since accelerated sharply. The reason is the modern reformulation of the equation in a (3+1)-dimensional, extended form that incorporates higher-order dispersion and anisotropic mixing of space–time derivatives [13]. This extended EPE has been shown to be Painlevé integrable, to admit multiple soliton solutions, and to support a remarkably broad spectrum of wave structures [13-15]. Crucially, it generalises classical low-dimensional models while retaining enough structure to permit systematic analytical treatment, which makes it an attractive bridge between abstract integrability theory and applied wave physics in plasmas, optical fibres, oceans, and fluids [15,16].

Crucially, the extended EPE is written in the literature in two closely related but structurally distinct ways, and the distinction is central to this review. The first is a fully polynomial, multiplicative formulation [13],

$$N_{tt} + (N_{xxt} - (3/4)(N_x^2 N_t)_x + 3 N_x N_t)_x + \alpha N_{xt} + \beta N_{yt} + \gamma N_{zt} = 0, \quad (1)$$

in which the nonlinear group is assembled from products of the field's derivatives; applying the product rule gives  $(N_x^2 N_t)_x = 2 N_x N_{xx} N_t + N_x^2 N_{tx}$ , so every term remains polynomial. The second is a rational, division formulation [14,15],

$$N_{tt} + (N_{xxt} - (3/4)(N_x^2 / N_t)_x + 3 N_x N_t)_x + \alpha N_{xt} + \beta N_{yt} + \gamma N_{zt} = 0, \quad (2)$$

in which the same nonlinear contribution is written as a quotient,  $N_x^2/N_t$ . The two forms coincide only when the field varies purely in space, and they are not equivalent in general: they differ in integrability, in singularity structure, and in physical interpretation. These differences are examined systematically in Section 4.

The present article reviews the EPE literature up to 2026, with a deliberate emphasis on the high-dimensional regime. Three questions organise the review. First, what is the mathematical status of the equation: where does it come from, and in what sense is it integrable? Second, which analytical methods have been applied, and how do they compare in scope, reliability, and computational cost? Third, what wave phenomena have been reported, and how well are the resulting solutions validated against physical requirements?

Answering these questions requires more than a catalogue of results. The field has expanded rapidly, and much recent output consists of method-driven solution construction that overlaps heavily with earlier work. A critical synthesis is therefore both timely and necessary. We pay particular attention to recurring methodological issues—redundant solution families, incomplete verification, and limited physical interpretation—and we identify the directions in which the field is genuinely advancing, namely variable-coefficient equations, fractional and stochastic generalisations, and the use of machine learning [17-19].

The remainder of the review is organised as follows. Section 2 states the review methodology, including search strategy, inclusion criteria, and the verification protocol applied to all references. Section 3 introduces the EPE and its mathematical structure. Section 4 analyses the structural and physical differences between its multiplicative and rational formulations. Section 5 examines integrability through Painlevé analysis and related tools. Section 6 provides a comparative survey of analytical methods. Section 7 classifies the resulting wave structures. Section 8 offers a critical analysis and identifies research trends. Section 9 surveys physical applications. Section 10 reviews fractional, stochastic, and data-driven extensions. Section 11 lists open problems and future directions, and Section 12 concludes.

## 2. Review Methodology

This review follows a transparent, reproducible protocol. The objective was to assemble, verify, and critically appraise the complete English-language research literature on the Estevez–Prada equation and its immediate relatives, with emphasis on high-dimensional formulations.

**Sources and search strategy.** Bibliographic records were retrieved from Crossref, OpenAlex, Semantic Scholar, arXiv, and Google Scholar. Searches combined the core term "Estevez–Prada" with the related names "Estévez", "Prada", and "Estevez–Mansfield–Clarkson", and with method terms such as "Painlevé", "Hirota", "Darboux", "Bäcklund", "Lie symmetry", "(G'/G)-expansion", "Jacobi elliptic", "extended direct algebraic", "soliton", "lump", "rogue wave", and "breather". Parallel searches targeted the high-dimensional context through terms including "(3+1)-



dimensional", "KP", "BKP", "Boussinesq", "nonlinear evolution equation", "fractional", "stochastic", and "physics-informed neural network". Searches were repeated with recent-date sorting to capture work published in 2025 and 2026.

**Eligibility.** Primary research articles, reviews, and method papers were included. Studies were eligible if they (i) analysed the EPE or a directly named relative, or (ii) introduced, refined, or applied an exact-solution method that has been used on the EPE family, or (iii) reported a high-dimensional nonlinear wave model or phenomenon that provides context for the EPE. Records that could not be resolved to a stable identifier were excluded. No language or journal restriction was imposed beyond English-language availability.

**Verification.** A defining feature of this review is that every cited work was verified programmatically. Each digital object identifier was resolved against the Crossref REST API, and the returned metadata—authors, title, journal, year, volume, issue, pages, and article type—were used to construct the reference list. Of the candidate records assembled, 133 resolved to complete, authentic Crossref entries and are cited here. This protocol eliminates the risk of fabricated or misattributed references, a recognised concern in rapidly growing application-oriented literatures.

**Time frame and scope.** The review covers the period from the introduction of the equation in 2004 through to work published in 2026, including studies available online ahead of print. Emphasis is placed on high-dimensional formulations, but the low-dimensional origins and the wider method literature are treated where they are necessary to interpret the (3+1)-dimensional results.

**Limitations.** Citation-based retrieval can under-represent very recent or non-indexed work. The review is qualitative: no meta-analysis was performed, because the reported solutions are expressed in incompatible parameterisations and rarely accompanied by comparable quantitative metrics. Comparisons of methods therefore rest on conceptual scope and reported solution families rather than on a common numerical benchmark.

### 3. The Estevez–Prada Equation and Its Mathematical Structure

The EPE emerged from a systematic effort to extend integrable equations into higher dimensions. Estévez and Prada first generalised the sine–Gordon equation to 2+1 dimensions, showing that the additional spatial variable could be accommodated without destroying the symmetry structure that underlies integrability [7]. They then developed the singular manifold method for a class of equations in 2+1 dimensions, providing a unified route to Bäcklund transformations, Lax pairs, and exact solutions [8]. A parallel line of work addressed integrable hierarchies and hodograph transformations for Camassa–Holm systems, where the interplay between nonlinear dispersion and nonlocal structure produces both smooth and singular travelling waves [9,10]. Estévez and Prada also gave algorithmic constructions of lump solutions in 2+1 integrable equations and studied discrete analogues through the same singular-manifold lens [11,20]. Together these contributions defined a family of models—including the equation that now bears their name—characterised by a delicate balance between dispersion and nonlinearity in more than one spatial dimension [12].

The modern focus of the field is the (3+1)-dimensional extended EPE. For a real-valued field  $N = N(x, y, z, t)$ , the governing equation is given by the multiplicative and rational forms in Eqs. (1) and (2) [13–15]. In both,  $\alpha$ ,  $\beta$ , and  $\gamma$  are non-zero real constants associated with the  $x$ ,  $y$ , and  $z$  directions, and three structural features deserve comment. The term  $N_{xxt}$  represents higher-order dispersion. The nonlinear group couples the field's derivatives and can be written either as the polynomial expression of Eq. (1) or as the quotient  $N_x^2/N_t$  of Eq. (2). These nonlinear terms balance dispersion and permit stable localised structures [13,14]. The mixed derivatives  $\alpha N_{xt}$ ,  $\beta N_{yt}$ , and  $\gamma N_{zt}$  encode anisotropic propagation: the three spatial directions enter the temporal evolution with different weights, so the equation is not rotationally symmetric [15].

This anisotropic, mixed-derivative structure is what makes the extended EPE a genuinely high-dimensional model rather than a relabelled one-dimensional equation. It is also what connects the EPE to a broader family of integrable multi-dimensional systems. The Kadomtsev–Petviashvili and B-type Kadomtsev–Petviashvili equations, the Boussinesq equation, the Ito equation, and combined pKP–BKP systems all share the same architectural logic: an integrable core in one dimension, extended by additional spatial derivatives and, frequently, additional nonlinear couplings [5,21–23].



The EPE is closely related to the Estevez–Mansfield–Clarkson (EMC) equation, which arises in the description of pattern formation in liquid drops and in shallow-water dynamics [24]. Because both carry the name of Estévez and address higher-dimensional shallow-water waves, the two are often studied together, and the methodological cross-pollination between them is substantial. The EMC equation has been analysed in fractional form [25], through Lie symmetry methods [26,27], and with a wide array of soliton techniques [16,28,29]. Time-varying-coefficient and multi-dimensional generalisations of the EMC equation have also been proposed, extending the same modelling philosophy that motivates the extended EPE [17,30,31].

#### 4. Structural and Physical Differences in the Estévez–Prada Equation

The two formulations in Eqs. (1) and (2) are superficially similar—they differ only in how a single nonlinear group is written—yet the distinction has structural and physical consequences that merit dedicated analysis.

**Canonical polynomial structure.** In the multiplicative formulation of Eq. (1), the nonlinear group is  $(N_x^2 N_t)_x$ . Expanding by the product rule gives  $(N_x^2 N_t)_x = 2 N_x N_{xx} N_t + N_x^2 N_{tx}$ , a sum of polynomial products. Substituting this back and applying the outer derivative with respect to  $x$  yields a partial differential equation that is polynomial in the field and its derivatives: no term contains the field in a denominator. Polynomial equations are the natural home of classical integrability theory [1]. They admit bilinear representations, they can be tested by the Painlevé method without the complications introduced by rational singularities, and they are the standard setting for Hirota's bilinear method and for travelling-wave reduction [32,33].

**Rational (division) structure.** In the division formulation of Eq. (2), the same nonlinear contribution is written as the quotient  $N_x^2/N_t$ . This single change has far-reaching effects. The equation is no longer polynomial: it contains a rational term whose denominator,  $N_t$ , vanishes whenever the field is momentarily stationary in time. At such points the equation is singular, and the solution space acquires a structure qualitatively different from the polynomial case—one associated with wave steepening, break-up, and blow-up rather than with smooth balance [14]. The standard bilinear formalism does not apply directly to a rational nonlinearity of this kind.

**Comparative breakdown.** Table 1 summarizes the principal differences between the two formulations across three domains: mathematical integrability, physical behaviour, and analytical tractability.

*Table 1. Structural and physical differences between the multiplicative and rational formulations of the Estevez–Prada equation.*

| Feature domain             | Multiplicative (polynomial) model                                                                              | Rational (division) model                                                                                                              |
|----------------------------|----------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|
| Mathematical integrability | Polynomial; consistent with Painlevé integrability; preserves multi-soliton collisions [13]                    | Contains the rational term $N_x^2/N_t$ ; introduces possible moving rational singularities; reported as passing the Painlevé test [14] |
| Physical phenomenon        | Stable multi-directional convection; coherent structures such as tsunami fields and ion-acoustic solitons [37] | Singular wave break or blow-up as $N_t$ tends to zero; wave-fracture zone at the singular boundary                                     |
| Analytical domain          | Polynomial PDE; amenable to Hirota bilinearisation and travelling-wave reduction [1,33]                        | Non-polynomial rational PDE; division-by-zero boundaries; resistant to direct bilinear mapping                                         |

**Origin of the division form.** How has a rational term come to appear in an equation that is canonically polynomial? The comparative analysis points to a methodological origin [14,15]. Protocols such as the  $(G'/G)$ -expansion method and the modified extended direct algebraic method reduce the nonlinear partial differential equation to an ordinary differential equation through a travelling-wave transformation, and then solve for the coefficients of an assumed series expansion using symbolic algebra [34–36]. The intermediate algebraic systems generated by computer-algebra systems such as Maple or Mathematica frequently contain rational expressions in the unknown coefficients. These rational



steps are an algorithmic device for solving the expansion, not a modification of the governing partial differential equation. When they are folded back into the model, they can give the impression that the underlying equation is rational rather than polynomial.

**Reconciling the two views.** The comparative picture outlined above must be held alongside the published literature, and the two do not yet agree. On the one hand, the polynomial formulation is the canonical form in integrability theory, and its Painlevé and bilinear properties are well established [1,32,33]. On the other hand, the (3+1)-dimensional extended EPE as extended by Wazwaz [13] and analysed by Datta and colleagues [14] is presented in the rational form, and a dedicated Painlevé analysis of that rational form concluded that it passes the Painlevé test and possesses integrable characteristics [14]. The apparent tension—a rational equation reported as integrable, in conflict with the polynomial criterion—is unresolved. We therefore treat the equivalence, or non-equivalence, of the multiplicative and division formulations as an open question and flag it as a priority for future work (Section 11). Settling it will require a matched asymptotic and singularity analysis of both forms under identical boundary conditions, and it would clarify which of the many reported solution families are genuine solutions of the canonical equation [13-15].

**Physical implications.** The practical stakes are significant. In the polynomial model, the nonlinearity supports stable, multi-directional coherent structures whose evolution is non-destructive—precisely the regime relevant to tsunami-type wave fields and to stable ion-acoustic solitons in plasmas [37,38]. In the rational model, the vanishing of  $N_t$  marks a singular boundary at which the field can break or blow up, so the same physical setting becomes associated with wave-fracture zones and abrupt collapse. This contrast is the clearest reason why the structural distinction matters for applications and not only for formal analysis.

## 5. Painlevé Integrability and Analytical Structure

Integrability is the property that most sharply distinguishes the EPE from generic high-dimensional NLPDEs. The standard diagnostic is the Painlevé test, which asks whether every singularity of a solution is a pole whose location is fixed by initial data—that is, whether the equation has no movable critical singularities [39-41]. An equation passing the test is a strong candidate for integrability and for possession of a Lax pair, an infinite set of conservation laws, and a rich family of explicit solutions [42].

The Painlevé analysis of the (3+1)-dimensional extended EPE has been carried out in recent work and confirms its integrability [13,14]. The analysis proceeds in three standard stages. In the leading-order step, the solution near a movable singular manifold is expanded and the dominant exponents are fixed. In the resonance step, the powers at which arbitrary functions enter the expansion are determined. In the final step, the resonance conditions are checked for consistency. For the extended EPE, the leading-order behaviour contains no critical branch points, and the resonance conditions are satisfied, so the equation passes the Painlevé test [14]. The same conclusion was reached independently through a Hirota-based proof of Painlevé integrability for the equation and its extensions [13].

This finding is important for two reasons. First, it justifies the search for the closed-form solutions that dominate the EPE literature: without integrability, such solutions would be exceptional rather than generic. Second, it situates the extended EPE within an established family of Painlevé-integrable higher-dimensional equations, including fifth-order extensions, KP-type and Hirota-type systems, and combined pKP–BKP models [43-47]. In each of these cases, Painlevé integrability is the property that licenses the construction of multi-soliton solutions and the derivation of Bäcklund transformations [48,49].

Beyond the Painlevé test, several related analytical tools have been applied to the EPE family and to comparable equations. The singular manifold method has been revisited and refined, and truncated Painlevé expansions now serve as a unified route to exact solutions and to the interaction of localised structures in (2+1) dimensions [42,50]. Painlevé analysis has also been used to derive higher-order rogue waves in generalised (3+1)-dimensional shallow-water equations, linking integrability directly to the existence of extreme-wave solutions [51]. Conservation-law analysis and nonlinear self-adjointness provide complementary evidence of structure: for a wide class of nonlinear wave equations, conservation laws can be constructed systematically once self-adjointness is established [52,53].



## 6. Analytical Methods for Exact Solutions: A Comparative Survey

The literature on the EPE and its relatives is, above all, a literature of methods. Each method imposes an ansatz on the solution, reduces the governing NLPDE to an ordinary differential equation (ODE) through a travelling-wave transformation, and solves the resulting algebraic system. The methods differ in the structure of the ansatz and, consequently, in the families of solutions they can reach. Table 2 summarizes the principal approaches and their trade-offs.

Table 2. Comparison of analytical methods applied to the Estevez–Prada equation

| Method                                      | Typical solution families                     | Key strengths                                 | Main limitations                                    | Representative refs |
|---------------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------------|---------------------|
| Hirota bilinearisation                      | Multi-soliton, rational, breather (limiting)  | Rigorous; proven interaction properties       | Reaches trigonometric/singular forms only by limits | [1,32,33,54]        |
| Darboux & Bäcklund transformations          | Soliton hierarchies, kink/anti-kink           | Exploits Lax pair; iterable                   | Needs a known seed or Lax pair                      | [57,59,61,63]       |
| Lie symmetry analysis                       | Invariant and similarity solutions            | Handles variable coefficients                 | Usually a smaller solution set                      | [64-66]             |
| tanh-coth & sine-cosine                     | Kink, soliton, periodic                       | Simple; computationally light                 | Restricted ansatz classes                           | [3,4,67]            |
| (G'/G)-expansion                            | Rational, hyperbolic, trigonometric           | Unifies several earlier expansions            | Redundant with direct methods                       | [35,36]             |
| MEDAM / EDAM                                | Hyperbolic, trigonometric, rational, singular | Broad; one algebraic system yields many forms | Large output; redundancy risk                       | [14,34,71]          |
| Auxiliary / Jacobi elliptic / exp-function  | Doubly periodic; soliton limits               | Systematic; generalises periodic waves        | Algebraically heavy                                 | [2,74-76]           |
| Inverse scattering / Riemann–Hilbert        | Full solution space from spectral data        | Gold standard for integrable PDEs             | High technical demand                               | [77-80]             |
| Bilinear neural networks / symbolic hybrids | Lump, breather, interaction solutions         | Handles complex multi-dimensional forms       | Less analytical insight                             | [81-84]             |

The Hirota bilinear method and its simplified variant transform the equation into a bilinear form and generate multi-soliton and rational solutions through a perturbation expansion [1,32,33,54]. It has been applied extensively to KP-type and related high-dimensional systems, and to multiple-soliton problems in fractional settings [55,56]. Its principal strength is rigour: because it builds on a bilinear representation, it yields soliton solutions with proven interaction properties. Its limitation is scope: it naturally produces exponential and rational solutions, and reaches trigonometric and singular forms only through limiting procedures.

Darboux and Bäcklund transformations take a different route. A Darboux transformation uses a Lax pair to map a known solution to a new one, and it can be iterated to generate hierarchies of solutions [57,58]. Recent applications include soliton management in variable-coefficient systems and non-degenerate solitons in coupled models [59,60].



Bäcklund transformations relate two solutions of the same or related equations and provide a route to auto-Bäcklund transformations, kink and anti-kink solutions [61-63]. For the Painlevé-integrable EPE family, auto-Bäcklund transformations have become a standard tool because they exploit the integrability directly [47].

Lie symmetry analysis identifies the continuous symmetries of the equation and uses them to reduce dimensionality and obtain invariant solutions [64,65]. It is especially valuable for variable-coefficient and higher-dimensional problems, where it can reveal the structural constraints that a model must satisfy to remain solvable [66]. However, symmetry reduction usually yields a smaller solution set than expansion methods.

Expansion methods form the largest group. The tanh-coth method and the sine-cosine method are among the earliest and most widely used, and they remain efficient for producing kink, soliton, and periodic solutions [3,4,67]. Their modern fractional and Camassa–Holm applications continue to appear [68,69]. The  $(G'/G)$ -expansion method and its two-function variant  $(G'/G, 1/G)$  generalise this family by expressing solutions through an auxiliary equation whose solutions are rational, hyperbolic, or trigonometric, thereby unifying several earlier techniques [35,36]. The direct algebraic method, from which these approaches descend, was formulated decades ago and remains the conceptual ancestor of the modern methods [70].

The modified extended direct algebraic method (MEDAM) is the most prominent recent addition. It assumes a finite Laurent-type series in an auxiliary function  $K(\phi)$  that satisfies a Riccati-type ODE, and it generates hyperbolic, trigonometric, and rational families according to the sign of the auxiliary equation's discriminant [34,71]. MEDAM has been applied to the extended EPE to obtain kink-type solitons, dark solitons, periodic oscillatory waves, and kink–antikink interactions [14]. Related direct algebraic and modified auxiliary-equation techniques have been applied to fractional Schrödinger, Ginzburg–Landau, and Cahn–Hilliard models, demonstrating the family's breadth [72,73]. Its attraction is combinatorial: a single algebraic system yields many solution forms. Its risk is the opposite of its strength—the large output invites redundancy and can obscure which solutions are genuinely new.

Auxiliary-equation and elliptic-function methods occupy adjacent territory. The auxiliary equation method and its descendants provide systematic ansätze whose solvability is guaranteed by construction [2,74]. Jacobi elliptic function expansions generalise hyperbolic and trigonometric solutions to doubly periodic ones, which degenerate to solitons and periodic waves in limiting cases [75]. Recent work continues to extend F-expansion and elliptic-function approaches to high-order and high-dimensional wave equations [76]. These methods are powerful but can be algebraically heavy, and the resulting solutions often require careful reduction before their physical meaning becomes clear.

Finally, inverse-scattering and integrability-based techniques address the equation at a deeper level. The inverse scattering transform, often formulated through a Riemann–Hilbert problem, recovers solutions from spectral data and is the gold standard for integrable equations [77-79]. Marchenko-type formulations provide a complementary route to soliton solutions when the spectral problem is non-standard [80]. Bell-polynomial and bilinear neural-network approaches provide alternative algebraic representations that can yield multi-soliton solutions for coupled systems [81,82]. Bilinear neural networks and related symbolic-numerical hybrids have been used to construct lump, breather, and interaction solutions for  $(2+1)$ - and  $(3+1)$ -dimensional equations [83-85]. These approaches are more demanding than expansion methods, but they connect the solution to the underlying integrable structure rather than to a chosen ansatz.

## 7. A Taxonomy of High-Dimensional Wave Phenomena

The methods of Section 6 generate a characteristic zoo of wave structures. Organising them into a taxonomy clarifies both the physical richness of the EPE and the recurring patterns of the literature.

Kink and anti-kink solutions are monotone fronts that connect two asymptotic states. They arise naturally from hyperbolic-tangent ansätze and from auto-Bäcklund transformations, and they are ubiquitous across KP-, BKP-, and pKP–BKP-type equations [21,63,86]. For the extended EPE, kink-type solitons were among the first structures reported [13,15], and kink–antikink interaction patterns have since been obtained, in which a kink and its mirror image propagate together [14]. Such configurations are physically relevant to phase transitions and domain-wall collisions in multi-dimensional media.



Bright and dark solitons are localised humps and depressions on a constant background, respectively. Dark solitons in particular appear in the extended EPE as smooth, symmetric U-shaped depressions that preserve their depth and shape during propagation [14]. This is notable because earlier studies of the EPE had reported primarily elevation-type structures, such as bright solitons and lumps [13,15]. The coexistence of elevation- and depression-type localised waves indicates that the equation supports both signs of nonlinear self-focusing, which broadens its physical applicability to optical and fluid systems where both polarities are observed [87].

Singular solutions form a distinct and often under-appreciated class. Because the extended EPE is rational in  $N_x^2/N_t$ , singular structures are intrinsic to the model. Recent work reports moving singularities—sharp, travelling divergences that propagate without spreading—and periodic trains of infinite spikes, both of which are absent from the Hirota-based and Jacobi-based solution sets that dominated earlier studies [14]. Singular solitons are sometimes dismissed as unphysical, but their systematic identification is valuable: they delineate the boundaries of the regular solution space and are relevant to extreme phenomena such as shock formation and rogue-wave precursors [51,88].

Lump solutions are rational, localised, and non-singular in two spatial dimensions. They were characterised for the KP equation and its generalisations [5,6], and their construction has been systematised through algorithmic and neural-network methods [11,84]. For the EPE family, lump solutions have been reported in the extended-Jacobi treatment of the equation [15] and in (3+1)-dimensional KP-type relatives [89-91]. Hybrid models in which a Korteweg–de Vries core is combined with Sawada–Kotera or Ramanie terms also support both lump and breather structures [92]. Rogue waves are anomalously large, transient localised waves that arise from modulation instability; they are well documented in the KP–BKP family and in shallow-water systems [51,88,93,94]. Breathers are localised in space and periodic in time, and they serve as building blocks for the Kuznetsov–Ma and Akhmediev solutions and for the Peregrine soliton in the limit [95]. Breather and hybrid breather–soliton solutions have been constructed for several high-dimensional equations closely related to the EPE [22,96,97].

Soliton molecules and multi-soliton bound states represent the most recent frontier. Initially observed in optics, where bound states of two or more solitons behave as a single particle [98,99], they have been extended to high-dimensional wave equations [89,100]. Their control and synthesis in laser systems illustrate the applied relevance of these structures [101,102]. For the EPE specifically, multi-solitonic forms have been reported using advanced ansätze, suggesting that the equation supports bound-state dynamics in three spatial dimensions [15].

Two structural patterns cut across this taxonomy. The first is the persistence of low-dimensional behaviour in higher dimensions: kinks, solitons, and periodic waves retain their one-dimensional character along a preferred direction, and the additional dimensions manifest as modulations rather than as qualitatively new structures. The second is the emergence of genuinely multi-dimensional objects—lumps, breathers, and complexiton-type solutions—that have no one-dimensional analogue [89,103]. The EPE, with its anisotropic mixed derivatives, supports both regimes, and the balance between them depends sensitively on the coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  [14].

## 8. Critical Analysis and Research Trends

A critical reading of the EPE literature reveals both genuine progress and systematic weaknesses. This section addresses the weaknesses directly, then documents the trends that are reshaping the field.

**Redundancy and novelty claims.** The dominant publication pattern is the application of a new or renamed expansion method to the same equation, followed by a claim of novel solution families. In practice, many of the reported families are equivalent under reparameterisation, or are limiting cases of solutions already in the literature [34,71]. The vocabulary of methods has diversified— $\tanh$ – $\coth$ ,  $(G'/G)$ ,  $(G'/G, 1/G)$ , exp-function, auxiliary equation, extended direct algebraic, modified extended direct algebraic—while the underlying algorithmic content has changed less than the proliferation of names suggests [2,36,70,74]. For the EPE specifically, the MEDAM results are genuinely broader than earlier Hirota- and Jacobi-based solution sets, particularly for singular and dark-soliton structures [14], but the increment should be judged by the new structures it yields, not by the count of formulas.

**Verification and validation.** A second weakness is incomplete verification. Solutions are frequently presented without a check that they satisfy the original equation, without boundary-condition analysis, and without stability assessment. The best recent work addresses this by combining solution construction with Painlevé analysis and by



verifying that the equation is integrable before interpreting the solutions [13,14]. Stability and modulation-instability analyses, where performed, add important physical content [68,72]. As a rule, however, the EPE literature reports mathematical solutions and leaves physical validation to future work.

**Parameter interpretation.** The coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  encode anisotropy, yet they are often treated as free parameters chosen to make a solution tractable. This obscures the physical meaning of the results. When the coefficients are interpreted as directional coupling strengths, the solutions acquire a clear reading as anisotropic wave packets, and the sensitivity of the wave profile to each coefficient becomes a testable prediction [14,15].

**Trends.** Three trends are clearly visible. First, the field is moving to higher dimensions and to variable-coefficient models, which relax translational symmetry and better represent inhomogeneous media [17,66]. Second, integrability is being combined with computational methods: PINNs and related neural solvers are increasingly used to learn solutions, discover parameters, and predict interactions that analytical methods cannot reach [19,104-108]. Third, the equation family is being generalised to fractional and stochastic settings, reflecting the prevalence of memory effects and environmental noise in real systems [18,109]. These three trends are not independent: variable coefficients, fractional derivatives, and stochastic forcing all break the idealised symmetry of the classical model, and all demand the numerical or data-driven techniques that are now becoming available. Qualitative techniques such as bifurcation analysis of the reduced travelling-wave equation provide a complementary way to classify the solution space and to anticipate the transitions between wave types [110].

A comparative synthesis of the principal EPE-specific studies is instructive. Wazwaz established Painlevé integrability and constructed multiple soliton solutions through the Hirota bilinear method, reporting kink, periodic, singular, and rational structures [13]. Dwivedi and colleagues applied an extended Jacobi elliptic function expansion, obtaining soliton, lump, kink, and interaction solutions that degenerate to trigonometric and hyperbolic forms in limiting cases [15]. Datta and colleagues applied MEDAM, extended the solution landscape to moving singularities, periodic spike trains, dark-soliton depressions, and kink–antikink pairs, and confirmed integrability through a dedicated Painlevé analysis [14]. The trajectory across these studies—from a bilinear core, through elliptic generalisations, to a systematic algebraic method with singular and dark solutions—illustrates the broader maturation of the field, in which each contribution widens the accessible solution space rather than merely renaming it.

## 9. Physical Applications Across Disciplines

The EPE is motivated by, and applies to, several physical domains. Its mixed-derivative, anisotropic structure makes it a natural model for wave propagation in media that are directionally dependent.

**Plasmas.** Ion-acoustic and other plasma waves are frequently modelled by high-dimensional integrable equations. Travelling and soliton solutions of generalised (3+1)-dimensional equations have been used to describe plasma wave dynamics, and DJKM-type equations have been analysed specifically in plasma-physics settings [37,38]. The EPE's kink and singular solutions are relevant to plasma sheaths and to abrupt density transitions [14].

**Optical fibres.** Nonlinear Schrödinger-type models govern pulse propagation in optical fibres, and their high-dimensional and variable-coefficient versions describe realistic transmission regimes [87,111]. The EPE's connection to Camassa–Holm hierarchies, and the presence of dark solitons in its solution set, make it relevant to fibres supporting both bright and dark pulses [9,14]. Soliton molecules, first observed optically, provide another point of contact [98,99].

**Fluids and oceans.** Shallow-water wave theory is the original setting of the EMC equation and of many KP-type models [24,112]. Lump, breather, and rogue-wave solutions of high-dimensional equations have been used to model extreme ocean waves and wave–wave interactions [93,94,113]. The extended EPE's dark solitons and periodic structures extend this descriptive range [14,15].

**Bose–Einstein condensates and nonlinear optics.** Matter-wave solitons in spinor condensates and droplet formation in nonlocal media are described by multi-dimensional equations with controlled dispersion [114,115]. These systems share with the EPE the combination of higher-order dispersion and nonlinear coupling, and techniques developed for one domain transfer readily to the other [17].



**Mathematical and engineering contexts.** Beyond physics, high-dimensional nonlinear wave models have been applied to gas bubbling, aortic wave dynamics, and other engineering problems [65]. Such applications illustrate the breadth of the high-dimensional nonlinear-wave toolkit, even when the EPE itself is not the model of record.

## 10. Extensions: Fractional, Stochastic, and Data-Driven Models

The most active growth areas for the EPE family lie beyond the classical integer-order, deterministic setting.

**Fractional models.** Fractional derivatives capture memory and nonlocal effects and are natural when transport is anomalous. Conformable, beta, M-truncated, and local fractional derivatives have all been used to construct soliton solutions of nonlinear evolution equations [116-118]. Fractional Schrödinger, Klein–Gordon, and Kadoma models have been treated with these operators [111,119-124]. The fractional EMC equation has been analysed directly with modified extended direct algebraic and improved (G'/G) methods, yielding kink, bright, and singular solutions together with stability and modulation-instability analysis [25,125]. These studies indicate that the EPE family admits a consistent fractional generalisation, although the choice of fractional operator materially affects the solutions and no consensus on the most physical choice has emerged.

**Stochastic models.** Real media fluctuate. Multiplicative white noise has been incorporated into optical-soliton models, and soliton solutions have been derived in the stochastic nonlinear Schrödinger and reaction–diffusion settings [18,109,126]. Jacobian-elliptic-function solutions, stability analysis, and chaotic behaviour have been examined for stochastic perturbed nonlinear Schrödinger equations [127-129], and noise effects on Kaup–Newell-type systems in birefringent fibres have been quantified [130]. The natural next step—and the one explicitly identified as a future direction by the EPE community—is the stochastic EPE with multiplicative noise, which would test the robustness of its solitons against environmental perturbation [14].

**Data-driven and numerical methods.** Physics-informed neural networks (PINNs) solve forward and inverse problems for PDEs by embedding the governing equation in the loss function [107]. They have been applied to integrable systems through Lax-pair-informed architectures [19], to soliton collisions in plasma models [104], and to higher-order nonlinear Schrödinger equations with external potentials [105]. Parameter discovery and data-driven soliton construction have been demonstrated for fourth-order AKNS and related equations [106,108]. For high-dimensional PDEs, adaptive-sampling and deep-learning solvers address the curse of dimensionality directly [131-133]. These methods are especially promising for the EPE because they can handle the rational nonlinearity and the anisotropic mixed derivatives without requiring a closed-form ansatz, and because they can learn which coefficient regimes support stable structures [17].

## 11. Open Problems and Future Directions

Several well-defined open problems emerge from this review.

First, **standardisation and benchmarking**. The field lacks a common benchmark for comparing solutions and methods. A shared repository of EPE solutions, with agreed parameter regimes and quantitative metrics such as amplitude, width, and stability, would allow genuine comparison and would curb redundant publication. Related to this, a stability and modulation-instability analysis of the extended EPE's dark, singular, and periodic solutions remains largely absent and is a high-value target [68,72].

Second, **systematic singularity and extreme-wave theory**. The rational nonlinearity of the EPE generates moving singularities and periodic spike trains that are only beginning to be catalogued [14]. A complete classification of the equation's singular structures, and a physical interpretation of the associated extreme events, would connect the EPE to rogue-wave theory in a rigorous way [51,93].

Third, **fractional and stochastic generalisation of the EPE itself**. Fractional and stochastic techniques have been developed extensively on neighbouring equations but not yet applied to the extended EPE in a systematic way [18,25,109]. Establishing which fractional operators preserve integrability, and how multiplicative noise deforms the soliton spectrum, are natural and tractable next steps [14].



Fourth, **integrability of variable-coefficient and multi-dimensional generalisations**. Time-varying coefficients and additional spatial dimensions break translational symmetry and may destroy integrability [17,66]. Determining the conditions under which Painlevé integrability survives, and identifying the resulting conservation laws, would clarify the boundary between integrable and non-integrable high-dimensional wave models [45,46].

Fifth, **data-driven discovery and hybrid methods**. PINNs and bilinear neural networks can discover coefficients, predict interactions, and bridge analytical and numerical descriptions [19,84,107]. Hybrid workflows that use Painlevé analysis to constrain a neural solver, or that use symbolic methods to seed a numerical study, are an underexplored and promising direction [106,108].

Sixth, **physical validation**. Ultimately, the value of the EPE's solution set depends on its correspondence with observable wave phenomena. Identifying the physical systems whose parameters map onto the EPE's coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$ —and testing the predicted anisotropic wave profiles against experiment or simulation—would move the field from mathematical construction to physical prediction [112,114].

## 12. Conclusions

The Estevez–Prada equation is a mathematically rich and physically motivated model of high-dimensional nonlinear wave propagation. Introduced as part of a programme to extend integrable systems into higher dimensions, its (3+1)-dimensional extended form is now established as Painlevé integrable and as the source of a broad spectrum of wave structures: kinks and anti-kinks, bright and dark solitons, singular and periodic-singular waves, lumps, rogue waves, breathers, and multi-soliton bound states. A diverse toolkit—Hirota bilinearisation, Darboux and Bäcklund transformations, Lie symmetry, and a family of expansion methods culminating in the modified extended direct algebraic method—has been brought to bear, with each method illuminating a different part of the solution space.

This review has argued that the field's principal strengths are its integrability and the breadth of its solution taxonomy, and that its principal weaknesses are redundancy, incomplete verification, and limited physical validation. The clearest trends point toward variable-coefficient, fractional, stochastic, and data-driven formulations, which relax the idealised assumptions of the classical model. All 133 sources cited here were verified against Crossref, providing a reliable evidentiary base for future work. The EPE is well positioned to serve as a benchmark for high-dimensional nonlinear wave theory; realising that potential will require the community to standardise its methods, validate its solutions, and connect its mathematics to measurable physics.

## Availability of Data

All reference metadata cited in this review were verified programmatically through the Crossref REST API. No new primary data were generated.

## Author Contributions

The review was conceived, researched, written, and compiled by the author.

## Conflict of Interest

The author declares no conflict of interest.

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