



The Navier–Stokes Equations: A Critical Review of Theory, Computation, Turbulence, and Data-Driven Frontiers

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Abstract

The Navier–Stokes equations describe the motion of viscous fluids and stand at the center of modern fluid mechanics. They also define one of the hardest open problems in mathematics, for the question of whether smooth three-dimensional solutions can develop singularities in finite time remains unresolved. This review provides a critical, evidence-based synthesis of the subject from its foundations in the work of Leray and Hopf through the mathematical, computational, and data-driven landscape of 2026. We first examine the equations and their physical content, then trace the theory of well-posedness: weak and strong solutions, regularity criteria, partial regularity, uniqueness, and nonuniqueness. We appraise what is known about possible singularities and the energy and attractor theory that governs long-time behavior. The statistical description of turbulence, from Kolmogorov scaling to anomalous dissipation, is then reviewed alongside its unresolved tensions. We evaluate the numerical toolbox — finite difference, projection, finite element, spectral, discontinuous Galerkin, lattice Boltzmann, and finite volume methods — with attention to stability, accuracy, and the pressure-velocity coupling that complicates incompressible computation. Modeling hierarchies from direct numerical simulation through large-eddy and Reynolds-averaged closures are assessed,

as are physics-informed neural networks and machine-learning-accelerated solvers that have reshaped the field since 2019. Applications in cardiovascular hemodynamics, geophysical flows, multiphase systems, combustion, and viscoelastic fluids illustrate both achievement and persistent gap. We close with a concrete research agenda, emphasizing the convergence of mathematical proof, high-resolution computation, and physical measurement.

Keywords: Navier–Stokes equations; weak solutions; regularity; turbulence; direct numerical simulation; large-eddy simulation; physics-informed neural networks; hemodynamics; Reynolds number.

1. Introduction

Fluid motion surrounds us, from blood in arteries to air over aircraft wings to convection in the mantle and ocean. Its governing equations, the Navier–Stokes equations, balance inertia, pressure, and viscous stress. For an incompressible Newtonian fluid with velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure p , they read

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0,$$

where ν is the kinematic viscosity and $\mathbf{f}$ a body force [1,2]. The incompressibility constraint encodes mass conservation, and the nonlinear advection term $(\mathbf{u} \cdot \nabla) \mathbf{u}$ is the source of nearly every difficulty that follows [3]. The ratio of inertial to viscous forces defines the Reynolds number $Re = UL/\nu$, which spans from creeping flows near $Re \ll 1$ to atmospheric and oceanic turbulence at $Re \sim 10^7$ and beyond [4,5].



The system was derived in the nineteenth century from the continuum stress principle of Cauchy, and its name honors Claude-Louis Navier and George Gabriel Stokes for independently formulating the viscous term [3]. A century of experiment confirms that the equations capture an enormous range of real flows, yet the same equations resisted all attempts at a complete mathematical theory. In 2000 the Clay Mathematics Institute named the question of global existence and smoothness of solutions to the three-dimensional incompressible Navier–Stokes equations one of its seven Millennium Problems [3,6].

This review takes stock of where the field stands in 2026. We are deliberately critical: what is proved, what is believed, what is computed, and what remains open. The intended audience includes mathematicians, computational scientists, and engineers, because the deepest open problems sit precisely at the boundaries between these communities [7,8].

1.1 Domain, boundaries, and dimensionless regimes

The equations must be supplemented by initial data and boundary conditions. The no-slip condition, which states that the fluid adheres to solid walls, is the classical choice and is appropriate for most macroscopic flows [9]. Free-slip, Navier-slip, and stress conditions appear in geophysical and microfluidic settings, and the mathematical theory is sensitive to which condition is imposed [10,11]. On unbounded domains, decay conditions at infinity replace the wall laws [1,12]. The Reynolds number organizes the physics: at low Re the nonlinearity is weak and solutions are smooth and unique for all time; at high Re the flow becomes turbulent, and the energy cascade transfers kinetic energy to scales where viscosity acts [4,5]. Because Re in practical flows can exceed 10^6 , no simulation can resolve all scales directly, and the entire enterprise of turbulence modeling rests on this separation [8,13].

1.2 Why the equations are hard

Three features combine to defeat elementary analysis. The nonlinear term is quadratic and loses one derivative; the incompressibility constraint is nonlocal, because the pressure is determined globally by $\nabla \cdot \mathbf{u} = 0$; and the system is only partially dissipative in three dimensions, in the sense that the nonlinearity can amplify enstrophy through vortex stretching [14,15]. The same features make the equations a prototype for the modern theory of nonlinear partial differential equations: semigroup methods, maximal regularity, energy methods, compensated compactness, and convex integration have all been forged on this problem [6,16,17]. Understanding what is essential and what is incidental in this difficulty is itself an active research program [7].

2. Well-Posedness: Weak and Strong Solutions

2.1 Leray–Hopf weak solutions

The natural function-space setting for the initial-value problem treats the velocity in L^2 with one derivative, and requires the pressure to be recovered from the divergence constraint. Leray proved in 1934 the global existence of weak solutions in the whole space that satisfy an energy inequality [1], and Hopf extended the construction to bounded domains [2]. These Leray–Hopf weak solutions are today the standard object of the theory: they exist for all time, for arbitrary L^2 data and forcing, and their kinetic energy is nonincreasing [18]. A modern, self-contained construction unifying the whole space, the torus, and bounded domains, and comparing Leray approximation, Faedo–Galerkin, time-semi-discretization, and Smagorinsky–Ladyzhenskaya regularization, is given by Berselli and Spirito [18]. Galdi clarified the delicate relation between “very weak” and Leray–Hopf solutions [19], while Dubois characterized uniqueness of certain Leray–Hopf solutions [20].

The weak formulation is not merely a technical convenience. It reflects a genuine limitation: the equations do not, on their own, provide enough regularity to justify the classical manipulations that lead to the energy identity. Leray’s construction therefore works with an energy inequality, and much of the modern theory concerns the gap between the inequality and the identity [20]. In the two-dimensional case the theory is essentially complete: global well-posedness holds, the energy identity is valid, and the resulting solutions are unique [18,21]. Three dimensions are different in kind, not merely in degree. The best available results either impose extra conditions on the data, reduce the strength of the solution concept, or add dissipation [17,22].



The central defect of the theory is that uniqueness and regularity of Leray–Hopf solutions in three dimensions remain open in general. Weak solutions may be nonunique, as first shown by Ladyzhenskaya in the Hopf class [23], and the modern convex-integration revolution established that nonuniqueness is not an artifact of poor definitions. Buckmaster and Vicol proved that weak solutions of the three-dimensional Navier–Stokes equations can be nonunique [16], and later work showed “wild” solutions whose singular sets are small in time [24]. Nonuniqueness extends to two-dimensional hypoviscous systems [25] and to stochastic settings [26,27]. These results do not contradict physics: the constructed solutions violate the energy inequality and are far from the energy class in which uniqueness questions are posed [16].

2.2 Strong solutions and local well-posedness

Strong solutions exist locally in time for sufficiently regular data and are unique. Fujita and Kato placed the theory on a firm analytical footing by proving local well-posedness in the scale-invariant Sobolev spaces $\dot{H}^{1/2}$ [15]. Kato extended the theory to strong L^p solutions and used them to recover weak solutions through the vanishing-viscosity limit [28]. The scale invariance of the equations endows these spaces with special status: every known well-posedness result holds in a space whose norm is invariant under the natural scaling, and failure of that scaling marks the edge of the theory [15,28]. Strong solutions persist as long as the Beale–Kato–Majda criterion holds, namely that the L^∞ norm of vorticity stays integrable in time [14].

Two structural remarks frame the whole theory. First, local well-posedness plus the blow-up alternative implies that a singularity, if it occurs, must be accompanied by an unbounded growth of some scale-invariant quantity [14,15]. Second, once data are large or the domain is unbounded, the good behavior of strong solutions may be lost even when global existence of weak solutions is guaranteed [12,29]. These two observations explain why research has concentrated on (i) regularity criteria that rule out the dangerous growth, and (ii) partial regularity results that limit where singularities can concentrate [6,30].

2.3 Regularity criteria and partial regularity

Since global regularity is open, a large literature asks how close solutions can come to a singularity. Prodi and Serrin initiated the modern theory of regularity criteria: if a suitable norm of the solution is finite, the solution is smooth. The foundational Serrin criterion states that $\mathbf{u} \in L_t^q L_x^p$ with $2/q + 3/p \leq 1$ and $p > 3$ implies regularity [6]. Weak- L^p refinements at the endpoint were established by Bosia, Pata, and Robinson [31], and Serrin-type criteria in terms of the gradient of velocity by Wolf [32]. Kozono and Sohr gave an early and influential regularity criterion in terms of $\nabla \mathbf{u}$ [33]. These criteria are sharp in the sense that the spaces are exactly those left invariant by the equation’s scaling [6].

The deepest regularity result is the partial regularity theory of Caffarelli, Kohn, and Nirenberg for “suitable” weak solutions [30]. It shows that the one-dimensional parabolic Hausdorff measure of the singular set is zero, meaning that singularities, if they occur, are rare. The theory has been extended to hyperdissipative and fractional operators, where higher dissipation order improves the result: Ożański obtained partial regularity for Leray–Hopf solutions of hyperdissipative equations [17], Chen established partial regularity for steady hyperdissipative fractional Navier–Stokes [22], and Chen and Wei treated the time-dependent fractional case [34]. Very recent work on hyperdissipative systems bounds the number of possible singular points [35], and the local regularity theory near curved boundaries has been refined in 2026 [36]. Regularity criteria in terms of individual velocity components or strain geometry continue to be sharpened [37,38].

2.4 The problem of finite-time blowup

Whether smooth solutions can lose regularity in finite time is the core open question. No proof of blowup exists, and none of global smoothness either. Tao showed that the difficulty is not an artifact of the exact equations by constructing an averaged variant of the three-dimensional Navier–Stokes system that provably develops finite-time blowup, even though it shares the energy structure and scaling of the true equations [7]. This result is a decisive warning: mechanisms of singularity formation that are allowed by the a priori estimates can indeed be realized in nearby models [7].

Numerical and analytical studies of the closely related three-dimensional Euler equations have produced strong evidence for singularity formation. Luo and Hou reported numerically stable, nearly self-similar blowup for the



axisymmetric Euler equations [39], with a comprehensive numerics guide following later [40], and Chen, Hou, and Huang constructed rigorous asymptotically self-similar blowup for the Hou–Luo model [41]. In 2025, Chen and Hou proved stable nearly self-similar blowup for the two-dimensional Boussinesq and three-dimensional Euler equations with smooth data [42]. Because vortex stretching is the same mechanism believed to drive Navier–Stokes singularities, these results sharpen the frontier: whether viscosity arrests the Euler-type mechanism remains unknown, but the gap has narrowed [14,42]. Simplified models of strain self-amplification also exhibit finite-time blowup [43]. On the other side, self-similar profiles that would describe a singularity are strongly constrained: Nečas, Růžička, and Šverák showed that nontrivial backward self-similar Leray solutions do not exist [44], a major obstruction to the most natural singularity ansatz. Numerically guided analysis continues to probe whether singularities can form in Navier–Stokes turbulence at all [45,46].

A parallel line of work studies models that lie between the full Navier–Stokes equations and analytically tractable systems. Hyperdissipative versions, in which the Laplacian is replaced by a fractional power $(-\Delta)^{\beta/2}$ with $\beta > 2$, smooth the small scales and make partial-regularity and even full-regularity questions more approachable; results there calibrate what the “correct” amount of dissipation buys [17,22,35]. Fractional and hypodissipative systems, by contrast, move in the opposite direction and exhibit new nonuniqueness phenomena [25,34]. The comparison between these models is one of the clearest available guides to the role of the Laplacian in three dimensions [7,17].

2.5 Related models and their lessons

The Navier–Stokes equations sit inside a web of neighboring systems whose study illuminates the original. The Euler equations, obtained by setting $\nu = 0$, are the inviscid limit and the natural arena for the Onsager theory of anomalous dissipation [47,48]. The Boussinesq equations couple velocity to a buoyancy field and share the vortex-stretching structure of Euler [42,49]. The Stokes equations, obtained by dropping the nonlinear term, are linear and perfectly well understood; they describe creeping flows and provide the pressure operator used throughout numerical analysis [50,51]. Chemotaxis–Navier–Stokes and magnetohydrodynamic couplings add fields that interact with the flow and change the balance of nonlinearity [11,45]. Boundary-layer (Prandtl) theory governs the thin region where no-slip conditions act, and its rigorous justification for the Navier–Stokes equations remains an active and delicate subject [52,53]. Each of these relatives sharpens our sense of which features of the three-dimensional problem are intrinsic [7,15].

3. Energy, Attractors, and Long-Time Behavior

The energy balance is central. Multiplying the equation by $\mathbf{u}$ and integrating shows that the rate of energy change equals the power input minus viscous dissipation [15]. For the unforced problem, energy decays monotonically, and for forced problems, the long-time dynamics is governed by the theory of global attractors. Foias and Temam estimated the Hausdorff dimension of the attractor of two-dimensional Navier–Stokes equations in terms of the Grashof number [21], and the finite-dimensionality of the long-time dynamics underpins reduced-order modeling [54,55]. In three dimensions, where uniqueness is open, the theory is subtle: Foias, Rosa, and Temam defined and studied the weak global attractor, whose topological structure is complicated precisely because of possible nonuniqueness [56]. Attractor dimension and small-length-scale estimates were connected by Gibbon and Titi [57], and Ziane obtained optimal bounds on the attractor dimension in certain geometries [58]. For flows past bodies and in exterior domains, large-time behavior is governed by the linearized Oseen dynamics, with decay rates and asymptotic profiles established by Miyakawa and Sohr [12] and refined in rotating-obstacle settings [59].

4. Turbulence

4.1 Kolmogorov’s theory

Fully developed turbulence is the statistical regime of the Navier–Stokes equations at very large Reynolds number. Its modern description begins with Kolmogorov’s 1941 theory, in which the energy injected at large scales cascades through an inertial range to the dissipation scale, where it is converted to heat [60,61]. The theory predicts that in the inertial range the energy spectrum follows $E(k) \sim \varepsilon^{2/3} k^{-5/3}$, where ε is the mean energy dissipation rate. Kolmogorov’s 1962 refinement, which incorporated the fluctuations of the dissipation rate and introduced intermittency corrections [4], anticipated the modern picture in which structure functions exhibit anomalous scaling



exponents [4]. The K41 phenomenology is presented critically in Frisch's monograph [62]. McComb and Fairhurst examined the status of the dissipation anomaly — the observation that the dissipation rate remains positive as viscosity tends to zero — against the K41 hypothesis of local stationarity [63].

Onsager's contribution connects turbulence to the mathematical theory. He conjectured that inviscid (Euler) solutions below a certain regularity threshold may dissipate energy even in the absence of viscosity [47]. Duchon and Robert made the Onsager conjecture precise by deriving an explicit formula for the anomalous dissipation of weak solutions [48], and Cheskidov, Constantin, Friedlander, and Shvydkoy proved the positive side of the conjecture for the Euler equations [64]. In the Navier–Stokes setting, anomalous dissipation has been established rigorously for forced systems [65,66], and Li, Yu, and Zhu obtained anomalous dissipation for the d -dimensional equations in 2025 [67]. Randomly forced systems show total dissipation beyond the deterministic bounds [26]. Physically, anomalous dissipation is the signature of the energy cascade: even as $\nu \rightarrow 0$, dissipation remains nonzero because the cascade transfers energy to ever-smaller scales [65].

4.2 Two-dimensional turbulence and beyond

In two dimensions, the enstrophy (integral of squared vorticity) is also conserved in the inviscid limit, which changes the direction of the cascade. Kraichnan predicted an inverse energy cascade to large scales and a forward enstrophy cascade to small scales [68,69]. Borue confirmed the inverse energy cascade in stationary two-dimensional homogeneous turbulence [70]. Two-dimensional and quasi-two-dimensional models have become essential tools for understanding geophysical turbulence [69].

The failure of strict K41 scaling is captured by the theory of intermittency. Structure functions of order p scale as $\langle |\delta u_\ell|^p \rangle \sim \ell^{\zeta_p}$ with exponents ζ_p that deviate from the K41 prediction $p/3$, and the deviations grow with p [4,62]. Multifractal and cascade models reproduce the observed curvature of ζ_p and connect intermittency to the spatial concentration of dissipation on a set of smaller dimension [62]. This phenomenology is not merely decorative: it quantifies how rare but intense events — the large velocity increments that dominate high-order statistics — shape transport, mixing, and the loading of structures [4,8]. The rigorous connection between intermittency and the possible blowup of solutions remains unresolved, though the local Reynolds-number argument provides a plausible bridge [45,46].

Constrained and simplified models have been used to isolate specific mechanisms of the turbulent cascade. She and Jackson introduced a constrained Euler system that reproduces the scaling of Navier–Stokes turbulence without dissipation [71], and a reversible constrained system was later shown to capture key statistical features of three-dimensional turbulence [72]. Modern high-resolution direct numerical simulation continues to refine our picture of the energy cascade and of small-scale intermittency [8,71], including the effects of finite arithmetic precision at extreme resolutions [73].

5. Numerical Methods

5.1 Finite differences and projection methods

The numerical solution of the incompressible Navier–Stokes equations is dominated by the difficulty of satisfying the divergence constraint while evolving the velocity. Chorin's projection method introduced the now-standard strategy of splitting the momentum equation from the constraint and projecting the intermediate velocity onto the space of divergence-free fields [9,74]. Viscous-splitting convergence was analyzed by Beale and Majda [75]. In the finite-volume tradition, Patankar and Spalding's SIMPLE algorithm and its successors couple velocity and pressure through a pressure-correction loop on staggered grids [76], and equal-order control-volume finite-element variants were developed by Prakash and Patankar [77]. A modern account of finite-volume methods for the Navier–Stokes equations is given by Perić [78]. Time integration of the unsteady incompressible equations, balancing accuracy and stability across splitting schemes, is surveyed by Marx [79].

Algebraic splitting methods place the projection idea in a linear-algebra framework. Saleri and Veneziani analyzed pressure-correction algebraic splitting for the incompressible Navier–Stokes equations [80], and the ALADINS solver couples algebraic splitting with adaptive time stepping [81]. Projection-based variational multiscale formulations



extend the approach to finite-element settings [82]. These methods remain the workhorses of incompressible flow simulation in engineering practice [76,77].

5.2 Finite element methods

The finite element method brings variational structure and geometric flexibility. Its foundation for incompressible flow is the mixed formulation in which velocity and pressure are sought in different finite element spaces, and stability requires the inf-sup (Ladyzhenskaya–Babuška–Brezzi) condition [51,83]. Girault and Raviart analyzed mixed finite element methods for the Navier–Stokes equations [83,84], and early combined finite element and marker-and-cell schemes extended the ideas to more general settings [85]. Le Tallec studied a mixed approximation with an emphasis on the nonlinear term [50]. Because standard equal-order elements violate the inf-sup condition, stabilized and variational multiscale methods were developed: John and Kaya introduced a finite element variational multiscale method for the Navier–Stokes equations with a full error analysis [82,86], and Badia and collaborators designed stabilized methods that circumvent the inf-sup condition through projection-based operators [87].

Adaptive finite element computation relies on a posteriori error estimation. Verfürth developed a posteriori estimators and adaptive mesh refinement for mixed finite element discretizations of the Navier–Stokes equations [88], and Prudhomme and Oden treated error estimation and control for the time-dependent equations [89]. Unconditionally stable schemes based on modified characteristics and other stabilized time-stepping approaches have been analyzed with optimal error estimates [90], and energy-stable schemes, including the scalar auxiliary variable (SAV) approach, guarantee unconditional stability for the incompressible equations at the discrete level [91]. The finite element approximation of the stationary and evolution problems continues to be refined through the framework of numerical analysis surveys [92,93].

5.3 Spectral, discontinuous Galerkin, and lattice Boltzmann methods

Spectral methods achieve exponential accuracy on simple domains and are the tool of choice for direct numerical simulation of homogeneous turbulence [8,71]. Bernardi, Maday, and Métivet analyzed spectral approximation of the periodic-nonperiodic Navier–Stokes equations [94], and He, Liu, and Sun developed multilevel spectral Galerkin methods with detailed error analysis in space and time [95,96]. These methods deliver the high resolution required for turbulence simulation [8]. Spectral discretizations also underlie many projection and splitting schemes used at scale [79,81].

A recurring theme across all discretization families is the need for discrete stability that mirrors the continuous energy balance. Unconditionally stable time-stepping methods, including scalar auxiliary variable (SAV) and modified characteristic schemes, guarantee that the discrete kinetic energy does not blow up regardless of the time step, and they come with optimal error estimates [90,91]. Pressure-robust formulations and projection-stabilized schemes address the separate issue that standard mixed methods can pollute the velocity error through the pressure [82,86,87]. Choosing among these options is a matter of matching the method to the regime: moderate Reynolds numbers favor robust finite elements with a posteriori control [88,89], while very high Reynolds numbers demand the efficiency and scaling of spectral, DG, or lattice Boltzmann codes [97–99].

Discontinuous Galerkin (DG) methods combine high order with geometric flexibility and local conservation. Hartmann and Houston derived an optimal-order interior penalty DG discretization of the compressible Navier–Stokes equations [97], Zhang constructed positivity-preserving high-order DG schemes that keep density and pressure positive [98], and multigrid solvers for high-order DG discretizations were developed by Shahbazi, Mavriplis, and Burgess [100].

Lattice Boltzmann methods offer a mesoscopic alternative in which the Navier–Stokes equations emerge from a discrete kinetic equation. Junk and Klar analyzed the discretization error of lattice Boltzmann schemes and their reduction to finite difference projection methods [99]. Models with flexible specific-heat ratio extend the method to compressible flows [101], entropic formulations guarantee stability at high Reynolds number [102], and finite-difference lattice Boltzmann schemes handle incompressible flows with simple boundary treatments [103].



6. From DNS to Engineering Models

Direct numerical simulation (DNS) resolves all scales of motion and requires resolution proportional to $Re^{9/4}$, confining it to moderate Reynolds numbers even on the largest computers [8,73]. DNS has nonetheless provided the benchmark data against which all models are judged [8]. Table 1 summarizes the resolution, cost, and role of the principal computational approaches.

Table 1. Principal computational approaches to the Navier–Stokes equations and their regimes of application.

Approach	What is resolved	Typical cost driver	Representative uses
Direct numerical simulation (DNS)	All scales	$Re^{9/4}$ grid points	Turbulence benchmarks, model calibration [8]
Large-eddy simulation (LES)	Energy-containing scales	Grid set by filter width	Unsteady engineering and geophysical flows [5]
Reynolds-averaged (RANS)	Mean flow only	Mesh for mean gradients	Design, optimization [13]
Hybrid and PANS	Fraction of energy spectrum	Bridging parameter	Wall-bounded flows at high Re [104]
Physics-informed learning	Data manifold + PDE residual	Training iterations	Inverse problems, sparse data [105]
Machine-learning surrogates	Learned corrections	Training data	Acceleration of repeated simulation [106]

The hierarchy is pragmatic. DNS is exact but unaffordable at high Reynolds number; LES and RANS trade fidelity for cost, and each introduces a closure problem whose resolution is empirical [5,13]. PANS and related bridging methods attempt to make the trade-off explicit through a resolution parameter [104]. The recent machine-learning literature can be read as an attempt to replace hand-tuned closures with data-driven ones, and as a new source of benchmark-quality surrogate models [105,106].

Large-eddy simulation (LES) resolves only the energy-containing scales and models the unresolved subgrid scales. Its origin is Smagorinsky's eddy-viscosity model, developed for numerical weather prediction and still the baseline of the field [5]. A broad family of eddy-viscosity and deconvolution subgrid models now exists [5]. Where LES is too costly and DNS impossible, Reynolds-averaged Navier–Stokes (RANS) models solve for the mean flow and model all turbulent fluctuations through closure hypotheses [13]. Partially averaged Navier–Stokes (PANS) models interpolate between RANS and DNS by resolving a chosen fraction of the kinetic energy [104], and hybrid RANS/LES methods are widely used in industrial and aerodynamic applications [107,108]. RANS closures remain indispensable in engineering design, from airfoils at low Reynolds number [107] to Rayleigh–Bénard convection [108], even as data-driven closures begin to compete with them [13,109].

7. Data-Driven and Machine-Learning Frontiers

Physics-informed neural networks (PINNs), introduced by Raissi, Perdikaris, and Karniadakis, embed the governing equations as soft constraints in the loss function and solve forward and inverse problems without classical discretization [105]. Physics-guided, physics-informed, and physics-encoded networks and operators have since been systematically organized into a hierarchy by Faroughi and co-workers [110]. Machine-learning-accelerated computational fluid dynamics, pioneered by Kochkov and collaborators, uses learned corrections to coarse-grid solvers to recover DNS-level accuracy at a fraction of the cost [106]. Applications of physics-informed learning in water systems, where data are sparse and physics is well known, were critically reviewed in 2026 [109], and physics-informed neural networks have been extended to viscoelastic creeping flows and non-Newtonian fluids [111].



Reduced-order modeling exploits the finite-dimensionality of the underlying dynamics. Proper orthogonal decomposition (POD) extracts the dominant modes from DNS or experiment, and closure strategies are needed because the truncated system is not closed [55]. Sensitivity-based improvements to POD models were introduced by Hay, Borggaard, and Pelletier [112]. POD reduced-order models now extend to the global ocean and atmosphere, where the state dimension is otherwise intractable [54,113]. Data assimilation, which synchronizes models to observations, has been shown to work for the Navier–Stokes equations at extreme Reynolds numbers by nudging toward resolved large scales [114].

It is important to state what the machine-learning literature has and has not achieved. Trained surrogates reproduce and accelerate dynamics on the data manifold on which they were trained [106], and physics-informed networks regularize learning with conservation laws when data are sparse or noisy [105,109]. What remains difficult is generalization: extrapolation in Reynolds number, geometry, and boundary conditions is not guaranteed by any of these methods, and physical consistency over long horizons is not enforced by pointwise loss functions alone [110]. Benchmarks, uncertainty quantification, and hybrid schemes that keep the discretized equations in the loop are therefore active topics of research rather than settled practice [106,110].

8. Applications

8.1 Cardiovascular hemodynamics

The Navier–Stokes equations are the foundation of computational hemodynamics. Patient-specific simulations of the carotid artery before and after stenting predict wall shear stress and hemodynamic alterations relevant to restenosis [115], and combined CFD–particle image velocimetry assessments of carotid hemodynamics quantify atherogenic risk in individual patients [116]. Cerebral aneurysm rupture risk is correlated with local hemodynamics, and patient-specific CFD studies quantify the influence of bleb morphology on intra-aneurysmal flow [117]. These applications require efficient solvers: parallel preconditioners for the unsteady Navier–Stokes equations enable realistic three-dimensional simulations of arterial flow [118]. Fluid–structure interaction extends the framework to deformable vessels and valves [119].

8.2 Geophysical and environmental flows

The Navier–Stokes equations underlie numerical weather prediction and ocean modeling. Coupled ocean–atmosphere general circulation models solve discretized forms of the primitive equations and drive climate science [120]. Large-scale oceanic currents can be recovered as shallow-water asymptotic solutions of the Navier–Stokes equations, connecting the full three-dimensional theory to reduced models of oceanography [121]. The primitive equations themselves descend from the Navier–Stokes equations in the thin-layer, hydrostatic limit [5,120].

8.3 Multiphase, combustion, and non-Newtonian flows

Multiphase flows require solving the Navier–Stokes equations for each phase together with interface dynamics. Navier–Stokes simulations capture the instability regimes of gas–liquid flows between inclined plates [122], the nonlinear wave regimes of falling films [122], and vertical sloshing under periodic excitation [123]. Diffuse-interface (phase-field) models couple the Navier–Stokes equations to a Cahn–Hilliard equation, and their Boussinesq approximation was analyzed by Vorobev [124]; structure-preserving discretizations of the coupled system were compared in 2026 [125]. In combustion, hybrid LES/RANS simulations of cavity-stabilized supersonic ethylene combustion show how Navier–Stokes solvers couple turbulent transport with chemical kinetics [126]. Non-Newtonian fluid–structure interaction, as for viscoelastic Oldroyd-B fluids in deformable channels, extends the framework to polymer processing and biomedical devices [119].

8.4 Low-Reynolds and micro flows

At low Reynolds number, inertia is weak but not always negligible. RANS simulations of low-Reynolds-number airfoils guide the design of small unmanned vehicles [107], and analysis of the Navier–Stokes equations in thin and confined geometries supports microfluidic device design [127].



9. Frontiers and Open Problems

9.1 The global regularity problem

The Millennium Problem remains open: no proof establishes either global smoothness or finite-time singularity for the three-dimensional incompressible Navier–Stokes equations [3,7]. Progress has nonetheless been substantial. Partial regularity guarantees that the singular set, if any, is small [30]. Regularity criteria restrict possible blowup [6,14]. Nonuniqueness results constrain what a resolution of the problem can look like [16,25]. Model problems have sharpened our understanding of singularity formation [7,42]. Stochastic variants of the equations, which model turbulence through random forcing, have their own well-posedness theory that is better developed than the deterministic one in important respects [10,11,127]. Recent work in 2026 continues to extend partial regularity near boundaries and for hyperdissipative systems [35,36].

9.2 The gap between mathematics and computation

A striking feature of the field is the gap between what mathematics can prove and what computation reveals. Numerically guided studies of the Euler equations point to singularity formation [39,40,42], while the best direct numerical simulations of Navier–Stokes turbulence have not found evidence of finite-time blowup in physical flows [8,73]. Whether this is because singularities do not occur, or because they require Reynolds numbers beyond reach, is unknown. High-precision DNS and asymptotically self-similar ansätze are the two tools most likely to settle the question [42,67]. The anomalous dissipation that connects the Euler and Navier–Stokes descriptions is now proved in simplified settings and observed numerically, but its complete mathematical characterization remains open [65–67].

9.3 Turbulence modeling in the age of machine learning

The modeling hierarchy of DNS, LES, and RANS remains pragmatic rather than principled [13,104]. Machine learning offers the prospect of data-driven closures trained on DNS and validated against experiments [106,109], and physics-informed approaches embed conservation laws directly in the learning objective [105,110]. The central challenges are generalization to unseen geometries and Reynolds numbers, uncertainty quantification, and physical consistency over long horizons [109,110].

9.4 Multiscale and multiphysics coupling

Real applications couple the Navier–Stokes equations to heat transfer, chemical reactions, phase change, elasticity, and electromagnetism [119,124,126]. Structure-preserving, thermodynamically consistent discretizations of such coupled systems are an active frontier [125], and reduced-order models that retain the dominant physics while discarding the computational cost are essential for design and control [54,55,113].

10. Conclusions

The Navier–Stokes equations have proved to be an inexhaustible source of mathematics and a remarkably reliable basis for engineering. Their theory has advanced from Leray’s existence proof to a detailed map of what could go wrong: partial regularity restricts singularities [30], criteria constrain their onset [6,14], and convex integration shows weak solutions can be wild [16]. Whether genuine singularities occur for the deterministic three-dimensional equations remains open, and model problems suggest the question is genuinely hard [7,42]. Computationally, the field now spans DNS of turbulence at extreme resolution [8], robust finite element and DG solvers with rigorous error control [82,97], and machine-learning-accelerated and physics-informed methods that are changing how simulation is done [105,106]. Applications from hemodynamics to weather and combustion demonstrate the equations’ reach. The next decade will likely be defined by the convergence of these threads: mathematical proof informed by computation, computation validated by physics-informed learning, and a theory of turbulence that finally reconciles Kolmogorov’s phenomenology with the rigorous theory of weak solutions [4,65].



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