International Journal of Creative and Open Research in Engineering and Management
ISSN: 3108-1754 (Online)
Volume 02 Issue 11 November-2026 | Impact Factor: 3.5



Digital Roots of Equivalent Successive Discounts: A Modular-Arithmetic Characterization

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Abstract—

Successive percentage discounts are usually evaluated by multiplying the fractions of the price that remain after each reduction. This paper examines a less familiar question: when does the exact equivalent discount have a digital root of 1? Using the standard discount formula and congruences modulo 9, we prove that one integer discount congruent to 1 modulo 9 guarantees this result even when the other discounts are terminating decimals. For sequences consisting entirely of integer percentages, we obtain a complete criterion: the product of the remaining-price factors must be divisible by 9. This occurs precisely when one factor is divisible by 9 or at least two factors are divisible by 3. Eleven integer discounts between 0% and 100% satisfy the first condition, and worked calculations illustrate both the criterion and its limits. The analysis provides a rigorous explanation of a useful answer-checking shortcut; it does not claim that the underlying digital-root identity is new. Exact arithmetic is essential, since rounding can change the result.

Keywords— Successive discounts; equivalent discount; digital root; modular arithmetic; congruence modulo 9; finite decimals.

How to Cite this Article:

Dubey, A. K. (2026). Digital Roots of Equivalent Successive Discounts: A Modular-Arithmetic Characterization. International Journal of Creative and Open Research in Engineering and Management, 02(11), 1-9. https://doi.org/10.55041/ijcope.v2i11.066

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https://doi.org/10.55041/ijcope.v2i11.066

I. INTRODUCTION

A sequence of percentage discounts reduces the price repeatedly: each new percentage is applied to the amount left after the previous reduction. For example, discounts of 10% and 20% on ₹1,000 leave ₹900 and then ₹720. The overall reduction is therefore 28%, not 30%. The same reasoning extends to any finite sequence of discounts.

The digital root of a positive integer is obtained by repeatedly adding its digits until one digit remains. Its connection with congruences modulo 9 follows from $10 \equiv 1 \pmod 9$. In aptitude problems, this observation can help check a calculated discount or eliminate an implausible answer [1], [4].

The question addressed in this study is more specific than the familiar digit-sum rule: which properties of the remaining-price factors force the equivalent discount to have digital root 1? The cited mathematical sources establish the general digital-root framework, but do not by themselves supply the discount-specific characterization developed here. This is a focused gap in the sources reviewed, not a claim that no related result exists elsewhere.

We first express successive discounts as a product, then establish a sufficient condition for terminating-



decimal discounts. For integer percentages, we derive an exact if-and-only-if criterion and illustrate it with examples, including a case where the criterion fails.

II. LITERATURE REVIEW

The modular interpretation of digit sums is well established. Izmirli [1] studies properties of digital roots, while Costa and coauthors [2] discuss iterated digit sums in a related number-theoretic setting. These works provide mathematical context rather than a formula specifically about successive discounts.

MathWorld [4] gives a concise account of digital roots. Examination-preparation material [3] provides an applied motivation for digit-sum checks, although it is not treated here as evidence of a new theorem. Our contribution is to connect the established modulo-9 principle to the product of the price fractions remaining after successive discounts.

III. METHODOLOGY

This is a theoretical study based on exact algebra and elementary number theory. No participants, surveys, experimental observations, or statistical datasets are involved. We derive the equivalent-discount expression, clear decimal denominators to obtain integers, and use divisibility modulo 9 to prove the results. The numerical examples are checks of the derived statements, not empirical validation.

Let $d_1, d_2, \ldots, d_n$ be percentage discounts applied successively to an initial price P. After all discounts, the remaining price is

$$P \times \prod_{i=1}^{n} ((100 - d_i)/100).$$

Taking P = 100, the equivalent single discount D, expressed as a percentage, is

$$D = 100\,[1 - \prod_{i=1}^{n} ((100 - d_i)/100)] = 100 - (\prod_{i=1}^{n} (100 - d_i))/100^{n-1}. \quad (1)$$

For two discounts a% and b%, equation (1) reduces to $D = a + b - ab/100$. Repeated use of this identity gives the same result regardless of grouping, because the remaining-price fractions are multiplied and multiplication is associative.

For a nonnegative terminating decimal, we define its digital root by deleting the decimal point from its exact decimal representation and taking the ordinary digital root of the resulting integer. Appending trailing zeros does not affect that value. The zero case is treated separately, with digital root 0.

IV. RESULTS AND DISCUSSION

Two results follow: a sufficient condition that allows other discounts to be terminating decimals, and a stronger characterization when every discount is an integer percentage.

Theorem 1 (Terminating-decimal discounts). Suppose $d_1, \ldots, d_n$ are percentage discounts with terminating decimal representations. If at least one is an integer x satisfying $x \equiv 1 \pmod 9$, then the exact equivalent discount has digital root 1.

Proof. Write $100 - d_i = q_i/10^{t_i}$, where t_i is a nonnegative integer and q_i is an integer. Put $T = \Sigma t_i$ and $K = T + 2(n - 1)$. From (1), $N = 10^K D = 100 \cdot 10^K - \prod q_i$ is an integer. For the designated integer discount x, choose $t_i = 0$, so $q_i = 100 - x \equiv 0 \pmod 9$. Hence $\prod q_i \equiv 0 \pmod 9$. Since $10 \equiv 100 \equiv 1 \pmod 9$, we obtain $N \equiv 1 \pmod 9$. Thus N, and therefore the exact terminating decimal D, has digital root 1. □

Corollary 1. The integer percentages 1, 10, 19, 28, 37, 46, 55, 64, 73, 82, and 91 are all congruent to 1 modulo 9. Any one of them, when included among successive discounts, guarantees digital root 1 for the exact equivalent discount.

Theorem 2 (Integer-percentage characterization). Let every d_i be an integer percentage and set $Q = \prod(100 - d_i)$. The exact equivalent discount D has digital root 1 if and only if 9 divides Q.

Proof. Equation (1) gives $100^{n-1}D = 100^n - Q$. The left side is an integer and, because multiplication by a power of 100 only shifts the decimal point, it has the same digital root as D. Modulo 9, the right side is congruent to $1 - Q$. Its digital root is 1 exactly when $1 - Q \equiv 1 \pmod 9$, or equivalently $Q \equiv 0 \pmod 9$. □

Equivalently, either one discount satisfies $d_i \equiv 1 \pmod 9$, or at least two discounts satisfy $d_i \equiv 1 \pmod 3$. Indeed, the product Q is divisible by 9 precisely when its factors collectively contain at least two factors of 3.

Worked examples. (1) For 10% and 20%, D = 28%; its digit sum is 10, so its digital root is 1. (2) For 19%, 12%, and 52%, D = 65.7856%, whose digit sum is 37. (3) For 10%, 81%, and 72.89%, D = 95.36419%, whose digit sum is 37. (4) For 4%, 13%, and 20%, D = 33.184%; although no discount is 1 modulo 9, two are 1 modulo 3, so the digital root is 1. (5) For 12%, 20%, and 30%, Q = 492,800 is not divisible by 9; D = 50.72%, whose digital root is 5.



Table I: Verification of Integer Discounts Congruent to 1 Modulo 9

Special discount	Other discounts	Equivalent discount and digital root
1%	25%, 33%	50.2525% → 1
10%	15%, 22%	40.33% → 1
19%	5%, 12%	32.284% → 1
28%	13%, 34%	58.6576% → 1
37%	11%, 42%	67.4794% → 1
46%	5%, 24%	61.012% → 1
55%	14%, 30%	72.91% → 1
64%	2%, 50%	82.36% → 1
73%	4%, 21%	79.5232% → 1
82%	13%, 7%	85.4362% → 1
91%	8%, 33%	94.4524% → 1

The table verifies all eleven qualifying integer discounts. Each exact equivalent discount has digital root 1, as predicted by the corollary.

In multiple-choice aptitude questions, test whether any discount is 1 modulo 9 or whether at least two are 1 modulo 3. If so, the exact answer must have digital root 1. Use this only to eliminate options; it may not distinguish options sharing that digital root.

The results assume exact arithmetic. Rounding an intermediate value or the final percentage may change its digit sum. The integer-percentage equivalence does not extend unchanged to arbitrary terminating decimals; Theorem 1 gives the relevant sufficient condition.

V. CONCLUSION

The remaining-price product links successive discounts with arithmetic modulo 9. For terminating-decimal discounts, one integer discount congruent to 1 modulo 9 is sufficient for digital root 1. For integer discounts, divisibility of the product of remaining-price factors by 9 gives the complete criterion. These results explain when the aptitude shortcut is valid and when it is not. Future work may examine other moduli and the effect of rounding.

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